Algebraic Statistics December 15-18, 2008

Serkan Hosten (SFSU), Lior Pachter (UCB), Bernd Sturmfels (UCB)

Algebraic Statistics in non-parametric Information Geometry

Giovanni Pistone

POLITECNICO DI TORINO

Thursday December 18, 2008

An example in Statistical Physics

- Ω is a finite sample space with N points.
- $U\colon \Omega \to \mathbb{R}_{\geq 0}$, $U(x) = 0$ for some $x \in \Omega$, $U \not\equiv 0$.

Gibbs model ...

$$p(x;\beta) = \frac{\mathrm{e}^{-\beta U(x)}}{Z(\beta)}, \quad Z(\beta) = \sum_{x \in \Omega} \mathrm{e}^{-\beta U(x)}, \quad \beta > 0.$$

- U is the *energy*, β is the *inverse temperature*, Z is the *partition function*, $\mathrm{e}^{-\beta U}$ is the *Boltzmann factor*.

... and its limits

As $\beta \to \infty$,

$$Z(\beta) \to \#\{x : U(x) = 0\}, \qquad \mathrm{e}^{-\beta U(x)} \to (x : U(x) = 0),$$

I.e. the weak limit of $p(\beta)$ as $\beta \to \infty$ is the uniform distribution on the states $x \in \Omega$ with zero energy.

Canonical variable, extended model

- Changing $U \to V = (\max U - U)$ and $\beta \to \theta = -\beta \in \mathbb{R}$ we get the same statistical model presented as an exponential model

$$p(x;\theta) \propto \mathrm{e}^{\theta V(x)}$$

- There are weak limits as $\theta \to \pm\infty$, the limits being the uniform distributions on the set of states that minimize or maximize the U function. Such limits are important in a number of applications, e.g. Statistical Physics or simulation methods in optimization. Therefore, the notion of closed or extended exponential model deserve much attention.
- A generic exponential model based on the *canonical statistics* V can be written

$$p(x;\theta) = \mathrm{e}^{\theta V(x) - \psi(\theta)} \cdot p(x)$$

where the canonical statistics itself is given up to an affine transformation.
- If a canonical variable is integer valued, we obtain a **toric model** for the likelihood p_θ / p.

Information geometry

- The exponential model
$$p(x;\theta) = \mathrm{e}^{\theta V(x) - \psi(\theta)} \cdot p(x)$$
has a number of interesting features such as the strict convexity of the cumulant function ψ or the relation $\psi'(\theta) = \mathrm{E}_\theta[V]$ which do not depend on the parametrization, but are related with the idea of representing the interior of the probability simplex with an affine space.
- In **non-parametric** Information Geometry the model is presented with respect to a reference density and the canonical variable is centered,
$$p(x;\theta) = \mathrm{e}^{\theta u(x) - \psi(\theta u)} \cdot p(x;0),$$
with $u = \theta(V - \mathrm{E}_{p_0}[V])$ and $\psi(\theta u) = \mathrm{E}_{p_0}[\mathrm{e}^u]$.
- This idea extends to the representation of a generic strictly positive density q in the form
$$q = \mathrm{e}^{u - \psi(u)} \cdot p(x)$$
where u is uniquely determined by the reference density p and by the condition $\mathrm{E}_p[u] = 0$.

IG is a family of manifolds on Δ

- From Amari work, we know that there are many (differential) geometries on the simplex of probability densities of a given sample space $(\Omega, \mathcal{F}, \mu)$.
- Let $\mathcal{M}_>$ denote the set of all positive densities of $(\Omega, \mathcal{F}, \mu)$. For each $p \in \mathcal{M}_>$ the mapping $s_p : q \to u$ is a chart. The atlas (s_p) defines the **e-manifold**
- The atlas of the charts $q \mapsto q/p - 1$ defines the **m-manifold**.
- According Amari, in between the e-manifold and the m-manifold there are other differential structures associated with the charts

$$q \mapsto \frac{\left(\frac{q}{p}\right)^\lambda - 1}{\lambda}$$

However, $\lambda^{-1}((q/p)^\lambda - 1)$ is bounded below by $-\lambda^{-1}$.
- Here, we discuss the construction of such geometries and their algebraic counterpart in the form of a generalization of the exponential case.
- S. Amari, H. Nagaoka, *Methods of information geometry* (American Mathematical Society, Providence, RI, 2000), ISBN 0-8218-0531-2, translated from the 1993 Japanese original by Daishi Harada

κ-exponential

G. Kaniadakis, based on arguments from Statistical Physics and Special Relativity, has defined the *κ-deformed exponential* for each $x \in \mathbb{R}$ and $-1 < \kappa < 1$ to be

$$\exp_\kappa(x) = \exp\left(\int_0^x \frac{du}{\sqrt{1+\kappa^2 u^2}}\right).$$

Note the special cases

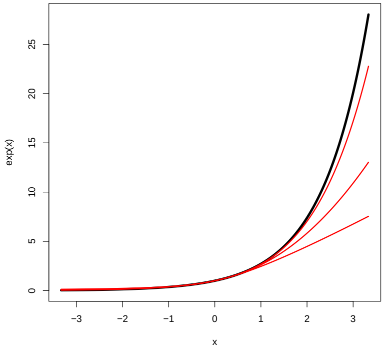
$$\exp_\kappa(x) = \begin{cases} \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}, & \text{if } \kappa \neq 0, \\ \exp x, & \text{if } \kappa = 0, \end{cases}$$

and the κ-deformed logarithm defined for $y > 0$ by

$$\ln_\kappa(y) = \begin{cases} \dfrac{y^\kappa - y^{-\kappa}}{2\kappa}, & \text{if } \kappa \neq 0, \\ \ln y, & \text{if } \kappa = 0. \end{cases}$$

- G. Kaniadakis, Physica A **296**, 405 (2001), G. Kaniadakis, Phisics Letters A **288**, 283 (2001);
- G. Kaniadakis, Physical Review E **66**, 056125 1 (2002), G. Kaniadakis, Physical Review E **72**, 036108 1 (2005).

Which deformation?



Among all possible approximations to exp, this particular one has been selected by Kaniadakis because it is the simplest with the property

$$\boxed{\exp_\kappa(x)\exp_\kappa(-x) = 1}$$

- For $\kappa \neq 0$, the indeterminate $y = (\exp_\kappa(x))^\kappa$ and x are related by the polynomial equation

$$\boxed{y^2 - 2\kappa x y - 1 = 0} \qquad \text{(HYP)}$$

- Therefore, the graph of $(\exp_\kappa)^\kappa$ is the upper branch of a hyperbola.

κ-deformed operations

- The function $\exp_\kappa$ maps $\mathbb{R}$ unto $\mathbb{R}_>$, it is strictly increasing and it is strictly convex.
- The function $\ln_\kappa$ maps $\mathbb{R}_>$ unto $\mathbb{R}$, is strictly increasing and is strictly concave.
- Both the κ-deformed exponential and the κ-deformed functions $\exp_\kappa$ and $\ln_\kappa$ reduce to the ordinary exp and ln functions when $\kappa \to 0$.
- Group operations $(\mathbb{R}, \overset{\kappa}{\oplus})$ and $(\mathbb{R}_>, \overset{\kappa}{\otimes})$ are defined in such a way that $\exp_\kappa$ is a group isomorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}_>, \overset{\kappa}{\otimes})$ and also from $(\mathbb{R}, \overset{\kappa}{\oplus})$ onto $(\mathbb{R}_>, \times)$:

$$\exp_\kappa(x_1 + x_2) = \exp_\kappa(x_1) \overset{\kappa}{\otimes} \exp_\kappa(x_2),$$
$$\exp_\kappa\left(x_1 \overset{\kappa}{\oplus} x_2\right) = \exp_\kappa(x_1) \exp_\kappa(x_2).$$

The algebra of $\exp_\kappa$ and $\ln_\kappa$

- The binary operations $\overset{\kappa}{\oplus}$ and $\overset{\kappa}{\otimes}$ are defined by

$$x_1 \overset{\kappa}{\oplus} x_2 = \ln_\kappa\left(\exp_\kappa(x_1)\exp_\kappa(x_2)\right)$$
$$y_1 \overset{\kappa}{\otimes} y_2 = \exp_\kappa\left(\ln_\kappa(y_1) + \ln_\kappa(y_2)\right)$$

- The operation $\overset{\kappa}{\otimes}$ is defined on positive reals. However, $\overset{\kappa}{\otimes}$ can be extended by continuity to non-negative reals in such a way that

$$0 \overset{\kappa}{\oplus} y = y \overset{\kappa}{\oplus} 0 = 0 \overset{\kappa}{\oplus} 0 = 0$$

- We want to derive defining relations for the κ-deformed operations in the form of a polynomial. This is obtained by repeated use of the HYP. Symbolic computations have been done with CoCoA.

- CoCoATeam, CoCoA*: a system for doing Computations in Commutative Algebra*, Available at `http://cocoa.dima.unige.it`.

- We want to find x such that $\exp_\kappa(x) = \exp_\kappa(x_1)\exp_\kappa(x_2)$.
- From $y_1 = (\exp_\kappa(x_1))^\kappa$, $y_2 = (\exp_\kappa(x_2))^\kappa$ and

$$(\exp_\kappa(x))^\kappa = (\exp_\kappa(x_1)\exp_\kappa(x_2))^\kappa = y_1 y_2,$$

  we have the ideal generated by

```
Eq1 := y[1]^2-2kx[1]y[1]-1;
Eq2 := y[2]^2-2kx[2]y[2]-1;
Eq3 := (y[1]y[2])^2-2kxy[1]y[2]-1;
```

- Elimination of y_1, y_2 gives the polynomial equation

$$\boxed{x^4 - 2\left(2\kappa^2 x_1^2 x_2^2 + x_1^2 + x_2^2\right)x^2 + \left(x_1^2 - x_2^2\right)^2 = 0},$$

  whose solution is

$$\boxed{x_1 \overset{\kappa}{\oplus} x_2 = x_1\sqrt{1+\kappa^2 x_2^2} + x_2\sqrt{1+\kappa^2 x_1^2}}.$$

- Kaniadakis has a relativistic interpretation.

- We want to find $z = \left(y_1 \overset{\kappa}{\otimes} y_2\right)^{\kappa}$. Let $y_1 = (\exp_\kappa(x_1))^\kappa$, $y_2 = (\exp_\kappa(x_2))^\kappa$, and $z = (\exp_\kappa(x_1 + x_2))^\kappa$.
- Equation HYP gives three quadratic equations in the indeterminates $x_1, x_2, y_1, y_2, z, \kappa$. Elimination of x_1, x_2 gives the polynomial equation

$$\boxed{y_1 y_2 z^2 + (1 - y_1 y_2)(y_1 + y_2)z - y_1 y_2 = 0}.$$

- It is remarkable that this equation does not depend on κ. An explicit solution is obtained by solving the quadratic equation.
- A possibly more suggestive solution is obtained as follows. First, we reduce to the monic equation

$$z^2 + \left(1 - \frac{1}{y_1 y_2}\right)(y_1 + y_2)z - 1 = 0$$

and denote the two solutions by $z > 0$ and $-1/z$. Therefore,

$$\boxed{z - \frac{1}{z} = \left(y_1 - \frac{1}{y_1}\right) + \left(y_2 - \frac{1}{y_2}\right)}$$

Box-Cox, Amari, generalised entropies

- The κ-logarithm is strictly related to a family of transformation which is well known in Statistics under the name of Box-Cox transformation or power transform. For data vector $y_1, \ldots, y_n$ in which each $y_i > 0$, the power transform is:

$$\boxed{y_i^{(\lambda)} \propto \frac{y_i^\lambda - 1}{\lambda}}$$

- The same transformation, applied to probability densities, appears in Amari as a device to construct Statistical Manifolds.
- Tsallis ha applied the transformation in non-extensive thermodynamics.
- Naudts discusses the applications of $\ln_\kappa$ and $\exp_\kappa$ in Information Theory and Statistical Physics.
- **Kaniadakis's κ-deformed logarithm $x = \ln_\kappa(y)$ has the extra feature of the symmetry induced by the term $-y^{-\kappa}$.**

- G.E.P. Box, D.R. Cox, J. Roy. Statist. Soc. Ser. B **26**, 211 (1964), ISSN 0035-9246.
- Monograph: S. Amari, H. Nagaoka, *Methods of information geometry* (American Mathematical Society, Providence, RI, 2000), ISBN 0-8218-0531-2, translated from the 1993 Japanese original by Daishi Harada.
- First paper: C. Tsallis, J. Statist. Phys. **52**(1-2), 479 (1988), ISSN 0022-4715.
- J. Naudts, Phys. A **316**(1-4), 323 (2002), ISSN 0378-4371; J. Naudts, JIPAM. J. Inequal. Pure Appl. Math. **5**(4), Article 102, 15 pp. (electronic) (2004), ISSN 1443-5756.

κ-Deformed Gibbs model I

- On a finite state space Ω, equipped with the energy function $U\colon \Omega \to \mathbb{R}_{\geq}$, we want to discuss the κ-deformation of the standard Gibbs model. **There are two options, related with two different presentation of the normalizing constant (partition function)**.
- The first option is to consider the statistical model

$$\begin{aligned} p(x;\theta) &= \frac{\exp_\kappa(\theta U(x))}{Z(\theta)} \\ &= \exp_\kappa\left(\theta U(x) \overset{\kappa}{\oplus} \ln_\kappa\left(\frac{1}{Z(\theta)}\right)\right) \end{aligned}$$

- The $\ln_\kappa$-model is, with $\widetilde{\psi}_\kappa(\theta) = \ln_\kappa Z(\theta)$,

$$\ln_\kappa p(x;\theta) = \theta U(x)\sqrt{1+\kappa^2(\widetilde{\psi}_\kappa(\theta))^2} - \widetilde{\psi}_\kappa(\theta)\sqrt{1+\kappa^2\theta^2 U(x)^2}$$

κ-Deformed Gibbs model II

- The second option is to define the model as

$$\begin{aligned} p(x;\theta) &= \exp_\kappa\left(\theta U(x) - \psi_\kappa(\theta)\right) \\ &= \exp_\kappa\left(\theta U(x)\right) \overset{\kappa}{\otimes} \exp_\kappa\left(-\psi_\kappa(\theta)\right), \end{aligned}$$

where $\psi_\kappa(\theta)$ is the unique solution of the equation

$$\sum_{x\in\Omega} \exp_\kappa\left(\theta U(x) - \psi_\kappa(\theta)\right) = 1.$$

- The derivative with respect to θ of ψ_κ is given by

$$\mathrm{E}_\theta\left[\frac{U - \psi'_\kappa(\theta)}{\sqrt{1 + \kappa^2\left(\theta U - \psi_\kappa(\theta)\right)^2}}\right] = 0,$$

where $\mathrm{E}_\theta\left[V\right] = \sum_x V(x)p(x;\theta)$.

Discussion

- The two one-parameter statistical models are different unless $\kappa = 0$. This fact marks an important difference between the theory of ordinary exponential models and κ-deformed exponential models.
- From the geometrical point of view, the second approach has the advantage of a the linear character of the model describing the $\ln_\kappa$-probability.

Let $V = \mathrm{Span}\,(1, U)$ and $V^\perp$ the orthogonal space, i.e. $v \in V^\perp$ if, and only if, $\sum_x v(x) = 0$ and $\sum_x v(x)U(x) = 0$. Therefore,

$$\sum_{x\in\Omega} v(x) \ln_\kappa (p(x;\theta)) = 0, \qquad v \in V^\perp$$

- Viceversa, if a strictly positive probability density function p is such that $\ln_\kappa p$ is orthogonal to $V^\perp$, then p belongs to the κ-Gibbs model for some θ.

κ-toric

- For each $v \in V^\perp$,

$$\sum_{x\colon v(x)>0} v^+(x) \ln_\kappa(p(x)) = \sum_{x\colon v(x)<0} v^-(x) \ln_\kappa(p(x))\,.$$

- A (physical) interpretation: a positive density p belongs to the κ-Gibbs model if, and only if,

$$\mathrm{E}_{r_1}[\ln_\kappa(p)] = \mathrm{E}_{r_2}[\ln_\kappa(p)]$$

  for each couple of densities r_1, r_2 such that $r_1 r_2 = 0$ and $\mathrm{E}_{r_1}[U] = \mathrm{E}_{r_2}[U]$.

If $v \in V^\perp$ happens to be integer valued, using the κ-algebra and the notation $\overbrace{x \overset{\kappa}{\otimes} \cdots \overset{\kappa}{\otimes} x}^{n \text{ times}} = x^{\overset{\kappa}{\otimes} n}$, we can write

$$\overset{\kappa}{\bigotimes}_{x\colon v(x)>0} p(x)^{\overset{\kappa}{\otimes} v^+(x)} = \overset{\kappa}{\bigotimes}_{x\colon v(x)<0} p(x)^{\overset{\kappa}{\otimes} v^-(x)},$$

Example 1/2

- $$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & U & v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -4 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & -1 & 1 \end{bmatrix} \end{array}$$
- The binomial equations are
$$\begin{cases} p(1) = p(2) \\ p(4) = p(5) \\ p(1) \overset{\kappa}{\otimes} p(2) \overset{\kappa}{\otimes} p(4) \overset{\kappa}{\otimes} p(5) = p(3)^{\overset{\kappa}{\otimes} 4} \end{cases}$$
- A non strictly positive density that is a solution is either $p(1) = p(2) = p(3) = 0$, $p(4) = p(5) = 1/2$, or $p(1) = p(2) = 1/2$, $p(3) = p(4) = p(5) = 0$. These two solutions are the uniform distributions on the sets of values that respectively maximize or minimize the energy function.

Example 2/2

- A further algebraic presentation is available. Consider the new parameters

$$\zeta_0 = \exp_\kappa\left(-\psi_\kappa(\theta)\right), \quad \zeta_1 = \exp_\kappa\left(\theta\right),$$

so that

$$\begin{aligned} p(x;\theta) &= \exp_\kappa\left(\theta U(x)\right) \overset{\kappa}{\otimes} \exp_\kappa\left(-\psi_\kappa(\theta)\right), \\ &= \zeta_0 \overset{\kappa}{\otimes} \zeta_1^{\overset{\kappa}{\otimes} U(x)}. \end{aligned}$$

The probabilities are κ-monomials in the parameters ζ_0, ζ_1, e.g.:

$$\begin{cases} p(1) = p(2) = \zeta_0 \\ p(3) = \zeta_0 \overset{\kappa}{\otimes} \zeta_1 \\ p(4) = p(5) = \zeta_0 \overset{\kappa}{\otimes} \zeta_1^{\overset{\kappa}{\otimes} 2} \end{cases}$$

- Note that the parameter ζ_0 is required to be strictly positive, while the parameter ζ_1 could be zero, giving rise the uniform distribution on $\{1,2\} = \{x \colon U(x) = 0\}$. The other limit solution is not obtained.

$\kappa \longrightarrow 0$

If $\kappa \neq 0$ the last equation of the system

$$\begin{cases} p(1) = p(2) \\ p(4) = p(5) \\ p(1) \overset{\kappa}{\otimes} p(2) \overset{\kappa}{\otimes} p(4) \overset{\kappa}{\otimes} p(5) = p(3)^{\overset{\kappa}{\otimes}4} \end{cases}$$

can be written as

$$\left(p^{\kappa}(1) - \frac{1}{p^{\kappa}(1)}\right) + \left(p^{\kappa}(2) - \frac{1}{p^{\kappa}(2)}\right) + \left(p^{\kappa}(4) - \frac{1}{p^{\kappa}(4)}\right) + \left(p^{\kappa}(5) - \frac{1}{p^{\kappa}(5)}\right) = 4\left(p^{\kappa}(3) - \frac{1}{p^{\kappa}(3)}\right)$$

Question

Is $\kappa \rightarrow 0$ a proper "approximation" of the regular case $\kappa = 0$?

κ-Divergence

- To construct an atlas, we define each chart as associated to a strictly positive probability densities. Such a density p is a reference for each other density q via the notion of likelihood q/p.

Definition

Fix a $\kappa \in]0,1[$. Given positive density functions q and p such that $\left(\frac{q}{p}\right), \left(\frac{p}{q}\right) \in L^{\frac{1}{\kappa}}(q)$, i.e. $\left(\frac{q}{p}\right)^{\kappa}, \left(\frac{p}{q}\right)^{\kappa} \in L^1(q)$, the κ-divergence is

$$D_\kappa(q\|p) = \mathrm{E}_q\left[\ln_\kappa\left(\frac{q}{p}\right)\right] = \frac{1}{2\kappa}\mathrm{E}_q\left[\left(\frac{q}{p}\right)^{\kappa} - \left(\frac{p}{q}\right)^{\kappa}\right].$$

- The strict convexity of $-\ln_\kappa$ implies

$$D_\kappa(q\|p) = \mathrm{E}_q\left[-\ln_\kappa\left(\frac{p}{q}\right)\right] \geq -\ln_\kappa\left(\mathrm{E}_q\left[\frac{p}{q}\right]\right) = \ln_\kappa(1) = 0.$$

with equality if, and only if $q = p$.

$\exp_\kappa$ densities

Definition?

$$\mathcal{E}_p = \left\{ q \in \mathcal{M}_> : \left(\frac{q}{p}\right)^\kappa, \left(\frac{p}{q}\right)^\kappa \in L^{1/\kappa}(p) \right\}$$
$$= \left\{ q \in \mathcal{M}_> : \frac{q}{p}, \frac{p}{q} \in L^1(p) \right\} = \boxed{\left\{ q \in \mathcal{M}_> : \frac{p}{q} \in L^1(p) \right\}}$$

- The divergence $D_\kappa(p\|q)$ is defined on $\mathcal{E}_p$.
- If $q \in \mathcal{E}_p$, then q is almost surely positive and we can write it in the form $q = \exp_\kappa(v) \cdot p$, with

$$v = \ln_\kappa\left(\frac{q}{p}\right) = \frac{\left(\frac{q}{p}\right)^\kappa - \left(\frac{p}{q}\right)^\kappa}{2\kappa} \in L^{1/\kappa}(p)$$

κ-exponential chart

p-chart $q \mapsto u$

The expected value at p of $v = \ln_\kappa\left(\frac{q}{p}\right)$ is $\mathrm{E}_p\left[\ln_\kappa\left(\frac{q}{p}\right)\right] = -D_\kappa(p\|q)$ so that we can write every $q \in \mathcal{E}_p$ as

$$q = \exp_\kappa\left(u - D_\kappa(p\|q)\right) \cdot p$$

where u is a uniquely defined element of the set of p-centered $1/\kappa$-integrable random variables $L_0^{1/\kappa}(p)$.

p-patch $u \mapsto q$

Vice versa, given $u \in L_0^{\frac{1}{\kappa}}(p)$, the real function $\psi \mapsto \mathrm{E}_p\left[\exp_\kappa\left(u - \psi\right)\right]$ is continuous and strictly decreasing from $+\infty$ to 0, therefore there exists a unique $\psi_{\kappa,p}(u)$ such that

$$q = \exp_\kappa\left(u - \psi_{\kappa,p}(u)\right) \cdot p \in \mathcal{E}_p \subset \mathcal{M}_>$$

Change of chart

Assume now we want to change of chart, that is we want to change the reference density from p_1 to p_2 to represent a q that belongs to both $\mathcal{E}_{p_1}$ and $\mathcal{E}_{p_2}$. The formal application of the chart and the patch formulæ gives

$$\begin{aligned} u_2 &= \ln_\kappa\left(\frac{q}{p_2}\right) - \mathrm{E}_{p_2}\left[\ln_\kappa\left(\frac{q}{p_2}\right)\right] \\ &= \ln_\kappa\left(\exp_\kappa\left(u_1 - \psi_{\kappa,p_1}(u_1)\right)\frac{p_1}{p_2}\right) - \mathrm{E}_{p_2}\left[\cdots\right] \\ &= \left(u_1 - \psi_{\kappa,p_1}(u_1)\right) \overset{\kappa}{\oplus} \ln_\kappa\left(\frac{p_1}{p_2}\right) - \mathrm{E}_{p_2}\left[\cdots\right] \end{aligned}$$

- Question: Is the set of u's such that $\exp_\kappa\left(u - \psi_{\kappa,p_1}\right) \cdot p_1$ belongs to $\mathcal{E}_{p_1}$ an open set of $L_o^{1/\kappa}(p)$?
- Problem: compute the Fréchet derivative of the change of coordinate.
- Problem: compute the connections.

Tangent vectors

- Let p_θ, $\theta \in]0,1[$, be a curve in $\mathcal{E}_p$,

$$p_\theta = \exp_\kappa\left(u_\theta - \psi_{\kappa,p}(u_\theta)\right) \cdot p.$$

- In the chart at p the velocity vector is given by

$$\dot{u}_\theta \in L_0^{1/\kappa}(p) = T_{\kappa,p}$$

- Formal computation gives

$$\frac{\dot{p}_\theta}{p_\theta} = (1 + \kappa^2(u_\theta - \psi_{\kappa,p}(u_\theta)^2)^{-1/2}(\dot{u}_\theta - D_{u_\theta}\psi_{\kappa,p}(\dot{u}_\theta))$$

so that

$$\boxed{\frac{\dot{p}_0}{p_0} = \dot{u}_0 - D_{u_0}\psi_{\kappa,p}(\dot{u}_0)}$$

Conclusion

- Amari tells us that each probability simplex Δ supports κ-statistical manifolds, one for each $\kappa \in [0, 1]$.
- Each κ has peculiar algebraic features.
- All κ-manifolds are possibly deduced from the same template, i.e. the exponential model (work in progress).
- There are domains of application of the algebro-geometric picture not yet explored:
  - Statistical Physics,
  - Optimization,
  - Differential equations for probability densities,
  - Approximation of statistical models.

THANKS