Interpolation

Starting point:

Given any

$$z_1, \ldots, z_{d+1} \in K$$

and

$$a_1, \ldots, a_{d+1} \in K,$$

there is a unique $f \in K[z]$ of degree at most d such that

$$f(z_i) = a_i, \quad i = 1, \ldots, d+1.$$

More generally:

Given any

$$z_1, \ldots, z_k \in K,$$

$$m_1, \ldots, m_k \in \mathbb{N} \quad \text{with} \quad \sum m_i = d+1,$$

and

$$a_{i,j} \in K, \quad 1 \leq i \leq k; \quad 0 \leq j \leq m_i - 1$$

there is a unique $f \in K[z]$ of degree at most d such that

$$f^{(j)}(z_i) = a_{i,j} \quad \forall i, j.$$

Problem:

What can we say along the same lines for polynomials in several variables?

First, introduce some language/notation. The "starting point" statement says that

$$H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \to \oplus K_{p_i} \to 0$$

or, equivalently,

$$h^1(\mathcal{I}_{\{p_1,\ldots,p_e\}}(d)) = 0$$

whenever $e \leq d+1$; more generally,

$$h^1(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)) = 0$$

when $\sum m_i \leq d+1$.

More generally, let $\Gamma \subset \mathbb{P}^r$ be a subscheme of dimension 0 and degree n. We say that Γ *imposes independent conditions* on hypersurfaces of degree d if the evaluation map

$$\rho : H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \to H^0(\mathcal{O}_\Gamma(d))$$

is surjective, that is, if

$$h^1(\mathcal{I}_\Gamma(d)) = 0;$$

we'll say it *imposes maximal conditions* if ρ has maximal rank—that is, if

$$h^0(\mathcal{I}_\Gamma(d))h^1(\mathcal{I}_\Gamma(d)) = 0.$$

Note that the rank of ρ is just the value of the Hilbert function of Γ at d:

$$\operatorname{rank}(\rho) = h_\Gamma(d);$$

and we'll denote it in this way in the future.

In these terms, the starting point statement is that *any subscheme of* $\mathbb{P}^1$ *imposes maximal conditions on polynomials of any degree.* Accordingly, we ask in general when a subscheme $\Gamma \subset \mathbb{P}^r$ may fail to impose maximal conditions, and by how much: that is, we want to

- characterize geometrically subschemes that fail to impose independent conditions; and

- say by how much they may fail: that is, how large $h^1(\mathcal{I}_\Gamma(d))$ may be (equivalently, how small $h_\Gamma(d)$ may be).

We will focus primarily on two cases: when Γ is reduced; and when Γ is a union of "fat points"—that is, the scheme

$$\Gamma = V(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k})$$

defined by a product of powers of maximal ideals of points.

As we'll see, these two cases give rise to very different questions and answers, but there is a common thread to both.

Note: we can ask similar questions for other classes of schemes Γ (for example, many results in the reduced case apply as well to curvilinear schemes), but it's unreasonable to ask about arbitrary schemes Γ, since we have no idea what these look like.

A. Reduced schemes.

In this case, the first observation is that *general points always impose maximal conditions*. So, we ask when special points may fail to impose maximal conditions, and by how much—that is, how small $h_\Gamma(d)$ can be.

Without further conditions, this is trivial: $h_\Gamma(d)$ is minimal for Γ contained in a line. It's still trivial if we require Γ to be nondegenerate: the minimum then is to put $n-r+1$ points on a line. So we typically impose a "uniformity" condition, such as linear general position.

We have then:

Theorem (Castelnuovo): If $\Gamma \subset \mathbb{P}^r$ is a collection of n points in linear general position, then

$$h_\Gamma(d) \geq \min\{rd + 1, n\}.$$

The proof is elementary; we exhibit hypersurfaces of degree d containing rd points of Γ and no others by taking unions of hyperplanes.

Moreover, this inequality is sharp: configurations lying on a rational normal curve $C \subset \mathbb{P}^r$ have exactly this Hilbert function.

The striking fact is the converse:

Theorem (Castelnuovo): If $\Gamma \subset \mathbb{P}^r$ is a collection of $n \geq 2r+3$ points in linear general position, and

$$h_\Gamma(2) = 2r + 1$$

then Γ is contained in a rational normal curve.

Thus we have a complete characterization of at least the extremal examples of failure to impose independent conditions.

The question is, can we extend this?

Conjecture

For $\alpha = 1, 2, \ldots, r-1$, if $\Gamma \subset \mathbb{P}^r$ is a collection of $n \geq 2r + 2\alpha + 1$ points in uniform position, and

$$h_\Gamma(2) \leq 2r + \alpha,$$

then Γ is contained in a curve $C \subset \mathbb{P}^r$ of degree at most $r - 1 + \alpha$.

"Uniform position" means: if $\Gamma', \Gamma'' \subset \Gamma$ are subsets of the same cardinality, then $h_{\Gamma'} = h_{\Gamma''}$

Notes:

1. The conjecture is known in cases $\alpha = 2$ (Fano; Eisenbud-Harris) and 3 (Petrakiev, [P])

2. This can't be extended as stated beyond $\alpha = r-1$: look for example at Γ contained in a rational normal surface scroll.

3. We know how to classify irreducible, nondegenerate subvarieties $X \subset \mathbb{P}^r$ with $h_X(2) = 2r+\alpha$. Thus all we have to do to prove the conjecture is to show that the intersection of the quadrics containing Γ is positive-dimensional.

4. A proof of the conjecture would yield a complete answer to the classical problem: for which triples (r, d, g) does there exist a smooth, nondegenerate curve $C \subset \mathbb{P}^r$ of degree d and genus g?

The bottom line:

Configurations $\Gamma \subset \mathbb{P}^r$ of points having small Hilbert function do so because they lie on small subvarieties $X \subset \mathbb{P}^r$—meaning, subvarieties with small Hilbert function. In this case, for small d the hypersurfaces of degree d containing Γ will just be the hypersurfaces containing X; in particular, X will be the intersection of the quadrics containing Γ.

Usually, to prove results along these lines it's enough to show the base locus $|\mathcal{I}_\Gamma(d)|$ is positive-dimensional.

B. Fat points.

Let $p_1, \ldots, p_k \in \mathbb{P}^r$ be general points, $m_1, \ldots, m_k \in \mathbb{N}$, and let

$$\Gamma = V(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}).$$

It is *not* always the case that Γ imposes maximal conditions on hypersurfaces of degree d!

So the first question is:

For what values of the integers r, k, $m_1, \ldots, m_k$ and d does Γ impose maximal conditions?

<u>Theorem</u> (Alexander, Hirschowitz): For $p_i \in \mathbb{P}^r$ general,

$$\Gamma = V(\mathcal{I}_{p_1}^2 \cdots \mathcal{I}_{p_k}^2)$$

imposes maximal conditions on hypersurfaces of degree d, with exactly four exceptions:

$r = 2$, $k = 2$, $d = 2$

$r = 2$, $k = 5$, $d = 4$

$r = 3$, $k = 9$, $d = 4$

$r = 4$, $k = 7$, $d = 3$

For general multiplicities m_i and general r, we don't even have a conjectured answer. For $r = 2$, though, we do. To express it, we introduce some more notation:

Let $p_1, \ldots, p_k \in \mathbb{P}^2$ be general, and let

$$S = \mathrm{Bl}_{\{p_1,\ldots,p_k\}}\mathbb{P}^2$$

be the blow-up of the plane at the p_i. Let ℓ be the divisor class of the preimage of a line in $\mathbb{P}^2$, and e_i the exceptional divisor over the point p_i.

Let L be the line bundle

$$\mathcal{O}_S(d\ell - \sum m_i e_i)$$

on S. Then

$$h^i(L) = h^i(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)).$$

In particular, the "expected dimension" of $h^0(L)$ is

$$\frac{(d+1)(d+2)}{2} - \sum \frac{m_i(m_i+1)}{2}$$

and this is exceeded exactly when the scheme Γ fails to impose independent conditions in degree d.

Conjecture (Harbourne-Hirschowitz):

Let S be the blow-up of $\mathbb{P}^2$ at k general points, L any line bundle on S. Then $h^1(L) \neq 0$ if and only there is a (-1)-curve $E \subset S$ such that

$$\deg(L|_E) \leq -2.$$

(Equivalently: if $h^1(L) \neq 0$, then the base locus of the linear system $|L|$ contains a multiple (-1)-curve.)

Notes:

1. If true, it gives a complete answer to our question for $r = 2$: conjecturally (more about this in a moment), we know where the (-1)-curves on S are, and can check the condition $\deg(L|_E) \leq -2$.

2. This is known for $k \leq 9$ (S has an effective anticanonical divisor).

3. This is known when $\max\{m_i\} \leq 7$ (S. Yang, [S])

A digression: my abysmal ignorance about curves on surfaces.

Let X be any smooth, projective surface, and consider the self-intersections of curves of X; that is, set

$$\Sigma = \{(C \cdot C) : C \subset X \text{ integral}\} \subset \mathbb{Z}.$$

Question: Is Σ bounded below?

I don't know the answer in characteristic 0, even for $X = S$ a blow-up of the plane at general points!

(János Kollár has an example in characteristic p; take Z a smooth curve of genus at least 2, $X = Z \times Z$, and $C_n \subset X$ the graph of the n^{th} power of Frobenius.)

By contrast, we can make a strong conjecture in the case of a blow-up S of $\mathbb{P}^2$ at general points.. Consider an arbitrary line bundle $L = \mathcal{O}_S(d\ell - \sum m_i e_i)$ on a general blow-up S. We have

The expected dimension of $H^0(L)$ is

$$\frac{(d+1)(d+2)}{2} - \sum \frac{m_i(m_i+1)}{2};$$

and the genus of a curve $C \in |L|$ is

$$\frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2}.$$

If we assume the first is positive and the second non-negative, it follows that the self-intersection of C is

$$(C \cdot C) = d^2 - \sum m_i^2 \geq -1.$$

Thus we may make the

Conjecture. Let S be a general blow-up of the plane, $C \subset S$ any integral curve. Then

$$(C \cdot C) \geq -1,$$

and if equality holds then C is a smooth rational curve.

If we believe this, the Harbourne-Hirschowitz conjecture should be equivalent to the weaker version:

Conjecture (HH; weak form):

Let S be the blow-up of $\mathbb{P}^2$ at general points, L any line bundle on S. If the linear system $|L|$ contains an integral curve, then $h^1(L) = 0$.

Conversely, if we believe the weak HH, it should be possible to prove the conjecture on self-intersections of curves on S, and thereby deduce strong HH.

Thus, I think it should be possible to prove that the two versions of HH are equivalent.

Notes on approaches to HH:

All approaches taken to HH (in case $k > 9$) involve specialization—Ciliberto and Miranda specialize a subset of the points p_i onto a line $L \subset \mathbb{P}^2$; Yang specializes the points onto a line one at a time.

Either approach involves an "apparent" loss of conditions; the goal is to understand what conditions the limit of the linear series $|\mathcal{I}_\Gamma(d)|$ will satisfy beyond the obvious multiplicity ones. These questions are fascinating in their own right.

An example: suppose $d = 4$, $k = 5$ and $(m_1, \ldots, m_5) = (1, 1, 1, 1, 3)$; suppose that $p_1, \ldots, p_4$ already lie on a line L and we specialize p_5 onto L.

The limits of the curves passing through $p_1, \ldots, p_4$ and triple at p_5 will be of the form $L+C$, with C a cubic double at p_5. But there are too many of these: cubics double at p_5 form a 6-dimensional linear system, while the system of quartics passing through $p_1, \ldots, p_4$ and triple at the general p_5 is only 4-dimensional.

So the question is: which cubics actually appear in the limiting curves?

The answer: cubics with a cusp at p_5, with tangent line L there.

It would be wonderful to understand better this limiting behavior. For example, does something like this occur when we specialize similarly defined linear systems on more general surfaces?

The bottom line:

There is one common thread running though our discussions of Castelnuovo theory and the Harbourne-Hirschowitz conjecture.

The content of the HH conjectures may be thought of as this: that if general multiple points in $\mathbb{P}^2$ fail to impose maximal conditions, they do so because they lie on a “small” curve—in this case, a curve of negative self-intersection.

The case $r \geq 3$:

As I said, we don't even have a conjectured answer to the question of when general multiple points impose independent conditions in higher-dimensional space. Based on our experience in $\mathbb{P}^2$, though, we might be led to make a qualitative conjecture:

Conjecture: Let $p_1, \ldots, p_k \in \mathbb{P}^r$ be general. If

$$h^1(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)) \neq 0$$

then the base locus of the linear series $|\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)|$ must be positive-dimensional.

Recasting the problem.

There is a common theme to our results and conjectures: we believe in many cases that when a subscheme $\Gamma \subset \mathbb{P}^r$ fails to impose independent conditions on hypersurfaces of degree d—that is, has small Hilbert function $h_\Gamma(d)$—it's because it's contained in a small positive-dimensional subscheme $X \subset \mathbb{P}^r$.

Moreover, X will appear as the intersection of the hypersurfaces of degree d containing Γ.

So let's recast the problem: let's drop all the conditions we've put on Γ at various points above, and instead make just one assumption: that the intersection of the hypersurfaces of degree d containing Γ is zero-dimensional; in other words, Γ *is a subscheme of a complete intersection of r hypersurfaces of degree d.*

We ask: what bounds can we give on $h^1(\mathcal{I}_\Gamma(d))$ (or $h_\Gamma(d)$, or $h^0(\mathcal{I}_\Gamma(d))$) under this hypothesis?

One further wrinkle: instead of specifying the degree n of Γ and asking for estimates on the size of $h^0(\mathcal{I}_\Gamma(d))$, let's turn it around: let's specify the dimension $h^0(\mathcal{I}_\Gamma(d))$, and ask for a bound on the degree of Γ.

Thus, we ask:

<u>Question</u>. Let $V \subset H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ be an N-dimensional linear system of hypersurfaces of degree d, with finite intersection Γ. How large can the degree of Γ be?

As a first example, let's try $d = 2$ and $N = r + 1$. The question is, in effect,

How many common zeroes can $r + 1$ quadrics in $\mathbb{P}^r$ have, if they have only finitely many common zeroes?

Or, to rephrase it,

Let $\{p_1, \ldots, p_{2^r}\} \subset \mathbb{P}^r$ be a complete intersection of quadrics in $\mathbb{P}^r$. How many of the points p_i can a quadric Q contain without containing them all?

Case $r = 2$: The answer is visibly 3.

Case $r = 3$: By Cayley-Bachrach, any quadric containing 7 of the 8 points of a complete intersection of quadrics in $\mathbb{P}^3$ contains the eighth as well; the answer is 6.

Case $r = 4$: Let $C = Q_1 \cap Q_2 \cap Q_3$. If two more quadrics had 13 common zeroes on C, they would cut out a g^1_3 on C. But C is not trigonal; the answer is 12.

Extrapolating, we might guess that

If $\Gamma = Q_1 \cap \cdots \cap Q_{r+1} \subset \mathbb{P}^r$, then

$$\deg(\Gamma) \leq 3 \cdot 2^{r-2}.$$

In fact, this has been proved by Robert Lazarsfeld ([L]) under the further hypothesis that Γ is reduced. Lazarsfeld actually answers the general question in case $N = r + 1$ under this hypothesis.

We do have a conjectured answer to the Question above; it's numerically complicated, but the underlying idea is straightforward. It is:

Conjecture. Among subschemes $\Gamma \subset \mathbb{P}^r$ lying on N hypersurfaces of degree d and cut out by them, the maximal degree is achieved by the scheme with ideal

$$I_\Gamma = (X_1^d, \ldots, X_r^d, F_1, \ldots, F_{N-r})$$

where $F_1, \ldots, F_{N-r}$ are the first $N-r$ monomials in lexicographical order, excluding the powers of the variables.

The actual degree of such a scheme Γ is hard to express as a function of r, d and N, but readily calculable in any given case. For more about this and related conjectures, see [EGH1], [EGH2] and [MP].

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