The geography of irregular surfaces

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Summary

Surface = smooth projective complex surface

Given a surface S, we denote as usual:

- $\mathcal{O}_S$, the *structure sheaf;*
- K_S, the *canonical divisor.*

The surface S is *of general type* iff K_S is **big**, i.e. if the linear system $|mK_S|$ is birational for $m >> 0$.

Surfaces not of general type are classified ("Enriques' classification"). They are divided in 7 classes.

Such a fine classification is not possible for surfaces of general type!

Surfaces of general type:

Definition: A surface S of general type is *minimal* iff K_S is **nef**.

- Every birational equivalence class of surfaces of general type contains precisely one minimal surface, called the *minimal model*.
- two minimal surfaces of general type are birational if and only if they are isomorphic.

So it is enough to study minimal surfaces up to biregular equivalence.

- Numerical invariants:
  - $p_g(S) := h^0(K_S) = h^2(\mathcal{O}_S)$, the *geometric genus;*
  - $q(S) := h^1(\mathcal{O}_S) = h^0(\Omega^1_S)$, the *irregularity* (recall: $q(S) = \frac{1}{2}b_1(S)$);
  - $\chi(S) := \chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$, the *holomorphic Euler characteristic*.
  - K^2_S, the self intersection number of K_S.

All these invariants, excepting K^2_S, are birational.
However, K^2_S is well defined if we take S minimal.

The numerical invariants are determined by the topology of S.
Usually $K^2_S, \chi(S)$ are chosen as the main numerical invariants.
Both invariants are known to be > 0.

Theorem (Gieseker 1977)

For every pair of positive integers (a, b) there exists a coarse moduli space $\mathcal{M}_{a,b}$ parametrizing minimal surfaces of general type with $K^2 = a$, $\chi = b$. The space $\mathcal{M}_{a,b}$ is quasi projective.

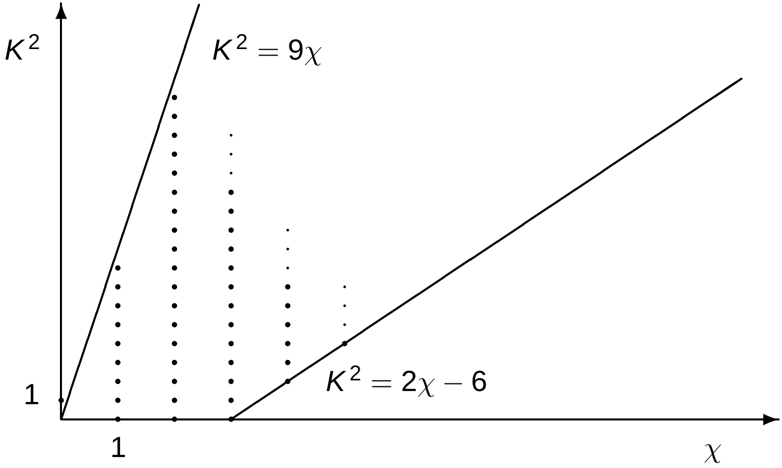
Question:

for what values of a, b is $\mathcal{M}_{a,b}$ nonempty?

namely, for what pairs (a, b) there exists a minimal surface of general type with $K^2 = a$, $\chi = b$?

Restrictions on K^2, χ:

- $K^2 > 0$, $\chi > 0$;
- $K^2 \geq 2\chi - 6$ (Noether's inequality);
- $K^2 \leq 9\chi$ (Bogomolov-Miyaoka-Yau inequality).


K^2
$K^2 = 9\chi$
$K^2 = 2\chi - 6$
1
1
χ

Surfaces on the "borders":

- Surfaces with $K^2 = \chi = 1$ ("Godeaux surfaces"): many examples are known and there is a partial classification.
- Surfaces on the line $K^2 = 2\chi - 6$ are classified (Horikawa, 1976). They exist for every value of $\chi \geq 4$ and are simply connected.
- Surfaces on the line $K^2 = 9\chi$ have the unit ball in $\mathbb{C}^2$ as universal cover (Yau's proportionality theorem).

Surfaces in the “interior”:

Persson (1981) has “filled” the region $2\chi - 6 \leq K^2 \leq 8\chi$, apart from a few exceptions. A great part of the remaning area has been then “filled” by Z.J. Chen (1988, 1991).

Sommese (1984) has shown that the possible ratios K^2/χ are a dense subset of the interval $[2, 9]$.

So it seems there are no “desertic areas" in the geography of surfaces.

A surface S is called *irregular* if the irregularity $q(S) = h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$ is > 0.

Remark: most of the examples by Persson and Chen are simply connected. In general, a lot of effort has been put into showing the existence of simply connected surfaces with all the possible invariants, while irregular surfaces have hardly been considered.

For an irregular surface, one defines an abelian variety $\mathrm{Alb}(S) := H^0(\Omega^1_S)^\vee / H_1(S, \mathbb{Z})$, the *Albanese variety*, and a morphism $a\colon S \to \mathrm{Alb}(S)$, the Albanese map.
The Albanese dimension $\mathrm{Albdim}(S)$ of S is by definition the dimension of $a(S)$.

An *irrational pencil* of genus $b \geq 1$ of S is a morphism with connected fibres $f\colon S \to B$, where B is a smooth curve of genus b.

If a surface has an irrational pencil, then of course it is irregular.

Remark: (Catanese) the existence of an irrational pencil of genus $b \geq 2$ is a topological property.

A surface has Albanese dimension 1 iff it has an irrational pencil of genus q. So the Albanese dimension is a topological property, too.

Question:

How do irregular surfaces fit in the geography of surfaces of general type?

More precisely, what are the restrictions on the invariants for:

- irregular surfaces?
- surfaces with an irrational pencil?
- surfaces with Albanese dimension 2?
- irregular surfaces without irrational pencils?

Irregular surfaces:

Surfaces on the Noether line $K^2 = 2\chi - 6$ are simply connected.

There are no irregular surfaces with $K^2 < 2\chi$.
The irregular surfaces with $K^2 = 2\chi$ have $q = 1$ and the fibres of the Albanese pencil have genus 2 (Horikawa).

Surfaces with an irrational pencil:

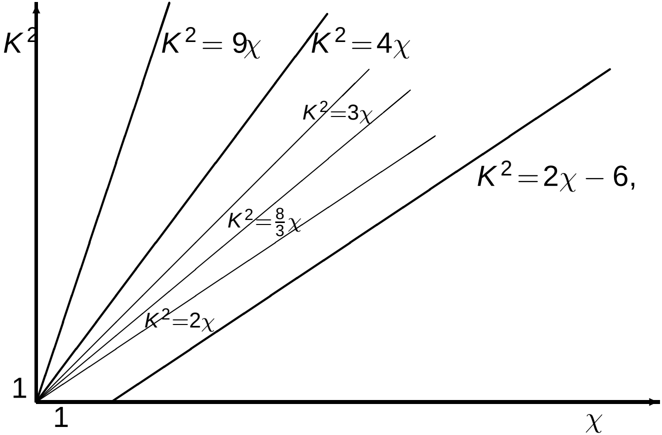
Slope inequality (Xiao Gang, Cornalba-Harris):

Let $f\colon S \to B$ be a relatively minimal fibration onto a smooth curve of genus b, with general fibre of genus g. If f is not smooth & isotrivial, then:

$$\frac{K_S^2 - 8(g-1)(b-1)}{\chi(S) - (g-1)(b-1)} \geq \frac{4(g-1)}{g}.$$

Corollary: surfaces with an irrational pencil of curves of genus g satisfy

$$K^2 \geq \frac{4(g-1)}{g}\chi.$$



As $g \to \infty$, the slope lines $K^2 = \frac{4(g-1)}{g}\chi$ converge to the line $K^2 = 4\chi$.

Surfaces with $\mathrm{Albdim} = 2$**:**

Theorem (Severi inequality):

Let S be a smooth minimal surface of general type. If $\mathrm{Albdim}(S) \geq 2$, then:

$$K_S^2 \geq 4\chi(S).$$

Some history:

- The inequality was first claimed by Severi in 1932, but his proof was wrong and it does not seem possible to fix it;
- it was rediscovered at the end of the '70's by Catanese, who realized that Severi's argument was incorrect, and by Reid, and stated as a conjecture;
- in 1987, Xiao Gang gave a proof for surfaces with an irrational pencil;
- Konno in 1993 proved it for surfaces with K_S divisible by 2;
- at the end of '90's, Manetti gave a proof under the assumption that K_S be ample;
- proved in full generality in 2004 (–).

Remarks & questions:

- The Severi inequality is sharp, but all the known examples with $K^2 = 4\chi$ have $q = 2$. This leads to the following

Conjecture (Reid, Manetti):

If $q \geq 4$ and Albdim $= 2$, then $K^2 \geq 4\chi + 4q - 12$.

- If the Albanese map is an immersion, except possibly at finitely many points, then $K^2 \geq 6\chi$. Can one get an inequality of the form $K^2 \geq c\chi$, with $c > 4$, under less restrictive assumptions on the Albanese map?

In order to discuss these questions, it is useful to compare the proof of the inequality and Manetti's proof for K_S ample. These proofs are completely different.

Sketch of proof (–):
$A := Alb(S)$, $\mu_d : A \to A$ multiplication by d. Fix H very ample on A. Note: $\mu_d^*(H) \sim_{num} d^2H$.
Have a cartesian diagram:

$$\begin{tikzcd} S_d \arrow[r, "\pi_d"] \arrow[d, "a_d"'] & S \arrow[d, "a"] \\ A \arrow[r, "\mu_d"] & A \end{tikzcd}$$

Set $L := a^*H$, $L_d := a_d^*H$.
The line bundle L_d is *nef & big* and $|L_d|$ is base point free.

Compute:

$$\chi(S_d) = d^{2q}\chi(S), \quad K_{S_d}^2 = d^{2q}K_S^2,$$

$$K_{S_d}L_d = d^{2q-2}K_S L > 0, \quad L_d^2 = d^{2q-4}L^2 > 0.$$

As d grows, L_d becomes "small" with respect to K_S.

Choose a general pencil in $|L_d|$ and blow up S_d to get a relatively minimal fibration

$$f_d : X_d \to \mathbb{P}^1.$$

with general fibre of genus

$$g_d = 1 + (K_{S_d} L_d + L_d^2)/2 = d^{2q-2} KL/2 + o(d^{2q-2}).$$

Note:

$$K^2_{X_d} = d^{2q} K^2_S + o(d^{2q}), \quad \chi(X_d) = d^{2q} \chi(S).$$

Write down the slope inequality for f_d:

$$\frac{K^2_{X_d} + 8(g_d - 1)}{\chi(X_d) + (g_d - 1)} \geq \frac{4(g_d - 1)}{g_d}.$$

The left hand side is equal to:

$$\frac{d^{2q}K^2_S + o(d^{2q})}{d^{2q}\chi(S) + o(d^{2q})}$$

Take the limit for $d \to \infty$:

$$\frac{K^2_S}{\chi(S)} \geq 4.$$

$\square$

Remarks:

- This proof tells us nothing about surfaces with $K_S^2 = 4\chi(S)$;
- it cannot be adapted to give a bound of the form $K^2 \geq 4\chi + aq + b$;
- given a refinement of the slope inequality, it might give a bound of the form $K^2 \geq c\chi$, with $c > 4$, under suitable assumptions on the Albanese map.

Outline of Manetti's proof:

Set $\mathbb{P} := \mathrm{Proj}(\Omega^1_S)$, $\pi\colon \mathbb{P} \to S$ the projection, $L := \mathcal{O}_{\mathbb{P}}(1)$.
Note: $L^2(L + \pi^* K_S) = 3(K_S^2 - 4\chi(S))$.
Assume for simplicity that there is no divisor on which every 1-form of S vanishes. Then the base locus of $|L|$ has dimension ≤ 1 and L^2 is represented by an effective 1-cycle Γ.
If $L + \pi^* K_S$ is nef, this finishes the proof.

Unfortunately this is not the case in general.

Decompose the cycle Γ representing L^2:
$\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$, where the cycles Γ_i are effective and
– $\pi(\Gamma_0)$ is contracted by a,
– $\pi(\Gamma_1)$ is in the ramification locus of a but is not contracted,
– Γ_2 is not in the base locus of $|L|$.
Manetti shows: $(L + \pi^* K_S)\Gamma_0$ can be < 0, but
$(L + \pi^* K_S)(\Gamma_0 + \Gamma_1 + \Gamma_2) \geq 0$.
This uses the connectedness of canonical divisors on a surface and a very fine analysis of the possible components of Γ_0. The assumption K_S ample is used here in an essential way.

$\square$

Remarks:

- It seems very difficult to adapt this method to the case when S has -2-curves.
- Under the assumption K_S ample, this proof yields a characterization of surfaces with $K_S^2 = 4\chi(S)$: they are double covers of abelian surfaces, branched on an ample divisor.
- it can be adapted to give a bound of the form $K^2 \geq 4\chi + aq + b$.

Theorem 1 (Mendes Lopes, – 2008):

Let S be a smooth surface of maximal Albanese dimension, with K_S ample and irregularity $q \geq 5$. Then:

$$K_S^2 \geq 4\chi(S) + \frac{10}{3}q - 8.$$

This is obtained by giving a lower bound for the term $(L + \pi^* K_S)\Gamma_2$ in Manetti's proof. We do this by a very careful study of the subsystem of $|K_S|$ generated by the divisors of the form $\alpha \wedge \beta = 0$, where α and β are 1-forms.
It is possible to get better bounds under extra assumptions on the Albanese map.

Theorem 2: (Mendes Lopes, – 2008):

Let S be a smooth surface of maximal Albanese dimension, with K_S ample and irregularity $q \geq 5$.

1. If the Albanese map $a\colon S \to A := \mathrm{Alb}(S)$ is not birational onto its image, then
$$K_S^2 \geq 4\chi(S) + 4q - 13;$$
2. if S has no irrational pencil and the Albanese map $a\colon S \to A$ is unramified in codimension 1, then
$$K_S^2 \geq 6\chi(S) + 2q - 8.$$

Remarks:

- We can give better inequalities when the canonical map is not birational.
- It is clear from the proofs that our results are not sharp: for fixed q or for $q >> 0$ one can give better inequalities, but it is very hard to give unified statements.

Surfaces with no irrational pencil of genus ≥ 2:

Theorem (Castelnuovo– de Franchis):

Let S be an irregular surface of general type. If S has no irrational pencil of genus ≥ 2, then

$$p_g(S) \geq 2q(S) - 3.$$

This can be rewritten as:

$$\chi(\omega_S) \geq (q - 2).$$

In this form the inequality has been recently extended to higher dimension:

Theorem (Pareschi–Popa 2008:)

Let X be a compact Kähler manifold with $\dim X = \operatorname{Albdim} X = n$. If there exists no surjective morphism $X \to Z$ with Z a normal analytic variety such that $0 < \dim Z = \operatorname{Albdim} Z < min\{n, q(Z)\}$, then:

$$\chi(\omega_X) \geq (q(X) - n).$$

Question:

What can one say about minimal surfaces with $p_g = 2q - 3$?

If $p_g = 2q - 3$ and S has an irrational pencil of genus ≥ 2, then there is a complete classification (Mendes Lopes – 2008, Barja-Naranjo-Pirola 2007): one gets either the product of two curves of genus 3 or a free $\mathbb{Z}_2$-quotient of a product of curves.

So, from now on, assume:
S minimal, $p_g(S) = 2q(S) - 3$ and S has no irrational pencil of genus ≥ 2.

Known properties of these surfaces:
Barja-Naranjo-Pirola have shown that if $|K_S|$ has no fixed component, then $K^2 \geq 8\chi$. Moreover we have:

Theorem 3: (Mendes Lopes – 2008):

- $K^2 \geq 7\chi - 1$;
- if $q \geq 7$ and $K_S^2 < 8\chi(S) - 6$, then the canonical map is birational;
- the Albanese map is birational onto its image;
- if $q \geq 5$, the canonical map has degree either 1 or 3.

What about the examples?

- If $q = 3$, then S is the symmetric product of a curve of genus 3 (Hacon –, Pirola-Catanese-Ciliberto-Mendes Lopes);
- There is no such surface with $q = 5$ (Pirola)
- If $q = 4$, then $K^2 = 16$ or 17 (Barja-Naranjo-Pirola, Causin-Pirola).

The difficulty in producing the examples is that the standard constructions either give surfaces with an irrational pencil or surfaces with $K^2 < 8\chi$ but $|K_S|$ free, and we have seen that this cannot happen for $p_g = 2q - 3$.

Conjectural example:

Let S be a minimal surface with $q = 4$, $p_g = 5 = 2q - 3$ and without irrational pencils.
If the canonical map φ of S has degree 2, then $K_S^2 = 16$, the canonical image $\Sigma \subset \mathbb{P}^4$ is the complete intersection of a quadric and a quartic. The map φ is a morphism, branched precisely on 40 nodes, which are the only singularities of Σ.

Conversely, given such a Σ with an even set of 40 nodes, the double cover of Σ branched over the nodes is a surface S as above.

Note: 40 is the maximum possible number of nodes for a complete intersection of a quadric and a quartic in $\mathbb{P}^4$. The code V corresponding to the even sets of nodes of Σ has length 40, dimension 8 and weights 16, 20, 24, 40. Such a code exists.

Question 1: do minimal surfaces with $p_g = 2q - 3$ satisfy the inequality $K^2 \geq 8\chi$?

Question 2: do surfaces with $p_g = 2q - 3$ and no irrational pencil have bounded invariants?