Deformations of Canonical Pairs and Fano varieties

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- A projective manifold X is **Fano** if $-K_X$ is ample (eg. $\mathbb{P}^N_{\mathbb{C}}$).
- We are motivated by the following result of Wiśniewski

Theorem

Let $f : X \to T$ be a smooth family of Fano manifolds. Then the Mori cone (of curves) $\overline{NE}(X) \subset N_1(X)$ and the the nef cone $Nef(X) \subset N^1(X)$ are locally constant.

Definitions

- Recall that $N_1(X) = \{C = \sum c_i C_i | c_i \in \mathbb{R},\ C_i \textit{ curve on } X\}/\equiv$ where $\equiv$ denotes numerical equivalence i.e. $C \equiv C'$ iff $(C - C') \cdot H = 0$ for any $\mathbb{R}$-Cartier divisor H.

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- Recall also that if A is ample, then the **nef threshold** $\tau_A(K_X) := \inf\{t \in \mathbb{R} | K_X + tA \text{ is ample}\}$.
- The above theorem follows from another result of Wiśniewski:

Nef thresholds in families

Theorem

Let $f : X \to T$ be a smooth family of projective manifolds and A an f-ample line bundle. If the nef threshold $\tau_{A|_{X_0}}(K_{X_0})$ is positive for some $0 \in T$, then the function $\tau_{A|_{X_t}}(K_{X_t})$ is constant for $t \in T$.

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- Note that since X_0 is Fano, its Mori cone is rational polyhedral and that for any extremal ray R, we may find a Cartier divisor D such that $R = \overline{NE}(X_0) \cap \{D_{=0}\}$. Since $-K_{X_0}$ is ample, we have $\epsilon D = K_{X_0} + A_0$ where $A_0 = \epsilon D - K_{X_0}$ is ample for $0 < \epsilon \ll 1$.

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- The proof of this results is of a topological nature, but nether-the-less it is natural to ask if it generalizes to the MMP context and if there are similar results for the pseudo-effective cone, the moving cone, the Mori chamber decomposition etc.

Outline of the talk

Singularities of the MMP

- A log pair (X, B) is a normal variety X and a $\mathbb{Q}$-divisor $B = \sum b_i B_i$ where $b_i \in \mathbb{Q}_{\geq 0}$ such that $K_X + B$ is $\mathbb{Q}$-Cartier.

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- (X, B) is canonical/terminal if the coefficients of F are $\leq 0 / < 0$ (for any Y).
- (X, B) is KLT if the coefficients of $F + \mu_*^{-1}B$ are ≤ 1.
- Warning: KLT sings are preserved under flips and divisorial contractions, but if (X, B) is canonical and $f : X \to X'$ is a divisorial contraction with $\mathrm{Ex}(f) \subset \mathrm{Supp}(B)$, then $(X', f_* B)$ is not canonical.

Deformations of singularities

- By results of Kawamata and Nakayama, it is known that:

Theorem

If $f : X \to T$ is a flat family over a smooth curve T and if the fiber X_0 has canonical/terminal singularities, then X_t also has canonical/terminal singularities for t is a neighborhood of $0 \in T$. If, moreover, K_X is f-big, then $P_m(X_t) = h^0(\mathcal{O}_{X_t}(mK_{X_t}))$ is constant.

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- The proof relies on a generalization of Siu's result on deformation invariance of plurigenera: Let $f : X \to T$ be a smooth family of projective varieties, then the functions $h^0(mK_{X_t})$ are constant.

Deformations of singularities II

- The idea of the proof is as follows: Asume that X_0 is canonical so that if $\mu : X' \to X$ is an appropriate resolution, we have $K_{X'_0} \geq \mu^* K_{X_0}$ where $X'_0 = \mu_*^{-1} X_0$.

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- Since X_0 is canonical, s lifts to a section of $\mathcal{O}_{X'_0}(mK_{X'_0})$ which (by a generalization of Siu's result) then lifts to a section of $\mathcal{O}_{X'}(m(K_{X'} + X'_0))$.

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- Pushing forward we have that the non-vanishing section s extends to a section $\tilde{s} \in \mathcal{O}_X(mK_X)$ and so K_X is $\mathbb{Q}$-Cartier.
- It also follows that $K_{X'} + X'_0 \geq \mu^*(K_X + X_0) \geq \mu^* K_X + X'_0$ so that X is canonical.

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- Variations on this example lead to examples were K_{Z_t} is not $\mathbb{Q}$-Cartier.
- If we restrict ourselves to the case of canonical singularities, then this problem does not occur.

Deformation of canonical pairs

Theorem

Let $S \subset X$ be an irreducible Cartier (in codimension 2) normal divisor on a normal variety X, $D \geq 0$ such that $S \not\subset \mathrm{Supp}(D)$. If $(S, D|_S)$ is canonical, then $K_X + S + D$ is $\mathbb{Q}$-Cartier near S and (X, D) is canonical near S.

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The proof follows either by results of Kollár-Mori, or by using extension theorems (to be discussed later) to show that sections of $m(K_S + D|_S)$ extend to sections of $m(K_X + S + D)$.

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- It follows that if $f : X \to T$ is a flat family and $(X_0, D|_{X_0})$ is canonical/terminal, then (X, D) and $(X_t, D|_{X_t})$ are canonical/terminal.

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- Assume that $(X_0, D|_{X_0})$ is terminal and $\mathbb{Q}$-factorial, $N^1(X/T) \to N^1(X_0)$ is surjective and $\psi : X \to Z$ is the contraction of a negative extremal ray of $N^1(X/T)$. If ψ_0 is a divisorial contraction that contracts no component of $D|_{X_0}$ (or if $-K_{X_0}$ is ψ-ample), then ψ and ψ_t are divisorial.

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- We use the last hypothesis to guarantee that $(Z_0, (\psi_0)_* D|_{X_0})$ (or K_{Z_0}) is canonical.

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- If $K_X + D$ is not nef then there is a curve $C \subset X$ such that $(K_X + D) \cdot C < 0$. By the Cone Theorem, we may assume that $R = \mathbb{R}^+[C]$ is an extremal ray of the cone of effective curves $\overline{NE}(X) \subset N_1(X)$ and that there is a contraction morphism $f = \mathrm{cont}_R : X \to Y$ such that $f_* C' = 0$ iff $[C'] \in R$.

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- If $\dim Y < \dim X$, then we have a log Fano fibration so that $-(K_X + D)$ is ample over Y. In particular $K_X + D$ is not in the closure of the cone of big divisors.

Running the MMP II

- If $\dim Y = \dim X$ and $\dim X = \dim \mathrm{Ex}(f) + 1$, then f is a divisorial contraction, (Y, f_*D) is KLT and we may replace (X, D) by (Y, f_*D).

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- If $\dim Y = \dim X$ and $\dim X > \dim \mathrm{Ex}(f) + 1$, then (Y, f_*D) is not KLT as $K_Y + f_*D$ is not $\mathbb{Q}$-Cartier. Instead we replace (X, D) by $(X^+, D^+ = \phi_*D)$ where the flip $\phi : X \dashrightarrow X^+$ is a birational map over Z which is an isomorphism in codimension 1 such that $K_{X^+} + D^+$ is KLT and ample over Y.

Running the MMP III

Theorem

([BCHM], [HM]) Let (X, D) be a klt pair such that D is big (or $K_X + D$ is big; or $K_X + D$ is not pseudo-effective).
Then there is a sequence of flips and divisorial contractions

$$X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N$$

such that either (X_N, D_N) is a minimal model (and $K_{X_N} + D_N$ is semiample) or there is a contraction morphism $f : X_N \to Z$ such that $-(K_{X_N} + D_N)$ is ample over Z.

The above sequence of flips and contractions is given by the MMP with scaling.

The MMP for families

- Let $f : X \to T$ be a flat family over a smooth curve such that $N^1(X/T) \to N^1(X_0)$ is surjective, D a divisor on X whose support does not contain X_0 and such that $(X_0, D|_{X_0})$ is a KLT pair with canonical singularities.

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- Assume that $D|_{X_0}$ (or $K_{X_0} + D|_{X_0}$) is big and that either:
  1. $\mathrm{SBs}(K_X + D)$ contains no component of $\mathrm{Supp}(D|_{X_0})$, or
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- If we run the $K_X + D$ MMP over T with scaling of some divisor H (in case 2 we assume that H is a multiple of $D - aK_X$), then we never contract a component of $\mathrm{Supp}(D|_{X_0})$ (or in case 2 each contraction is K_X negative).

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- By the previous result, this induces a $K_{X_0} + D|_{X_0}$ MMP$/T$.

An extension Theorem

- It follows that with the above assumptions:

Theorem

If L is an integral Weil $\mathbb{Q}$-Cartier divisor, $X_0 \not\subset \mathrm{Supp}(D)$ and $L|_{X_0} \equiv k(K_X + D)|_{X_0}$ for some $k \in \mathbb{Q}_{>1}$, then $H^0(X, \mathcal{O}_X(L)) \to H^0(X_0, \mathcal{O}_{X_0}(L))$ is surjective.

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- Proof: Let $\psi : X \dashrightarrow X'$ be a minimal model for $K_X + D$ and hence for $K_{X_0} + D|_{X_0}$. Then $H^0(\mathcal{O}_X(L)) \cong H^0(\mathcal{O}_{X'}(\psi_* L))$ and $H^0(\mathcal{O}_{X_0}(L)) \cong H^0(\mathcal{O}_{X'_0}(\psi_* L))$. The assertion now follows from Kawamata-Viehweg Vanishing.

Nef values in families

- We conjecture that Wiśnieski's result generalizises to the above situation:

Conjecture

Assume that $\mathrm{SBs}(K_X + D)$ *contains no component of* $\mathrm{Supp}(D|_{X_0})$ *(or* $D - aK_{X_0}$ *is ample for* $a > -1$*). If* $\tau_A(K_{X_0} + D|_{X_0}) > 0$ *for some ample line bundle* A*, then the function* $\tau_A(K_{X_t} + D|_{X_t})$ *is constant for* t *in a neighborhood of* $0 \in T$*.*

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- If $f|_{X_0}$ is of fiber type, then by semicontinuity, so is $f|_{X_t}$.
- If $f|_{X_0}$ is divisorial, then by our previous discussion, so is $f|_{X_t}$.

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- If $f|_{X_0}$ is small, the conjecture holds if $\dim X_0 \leq 3$ or if X_0 satisfies the volume criterion for ampleness that is, for any class $\xi_0 \in N^1(X_0)$, ξ_0 is ample iff for any ξ near ξ_0 we have $\mathrm{Vol}(\xi) = \xi^{\dim X_0}$.

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- This property is known to hold if X_0 is toric. Lazarsfeld raised the question of understanding for which other classes of divisors and of projective varieties it holds.
- Suppose that this property holds and that $\tau_A(K_{X_0} + D|_{X_0}) > \tau_A(K_{X_t} + D|_{X_t})$. Then there is a $\tau > 0$ such that $K_{X_t} + D|_{X_t} + \tau A|_{X_t}$ is ample but $K_{X_0} + D|_{X_0} + \tau A|_{X_0}$ is not nef.

Nef values in families III

- Let $\xi \in N^1(X/T)$ be sufficiently close to the class of $K_X + D + \tau A$. Then (using the extension theorem and the fact that $\xi|_{X_t}$ is ample)

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- Therefore X_0 does not satisfy the volume criterion for ampleness. Contradiction.

Outline of the talk

Moving and pseudo-effective cones

- Let $f : X \to T$ be a flat projective family such that X_0 is a Fano variety with $\mathbb{Q}$-factorial terminal singularities. By what we have already seen, for t in a neighborhood of $0 \in T$, X_t are also Fano varieties with $\mathbb{Q}$-factorial terminal singularities.

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- Note that given any divisor D, for $0 < \epsilon \ll 1$ we may write $\epsilon D \sim_{\mathbb{Q}} K_X + B$ where (X, B) is KLT and $B \sim_{\mathbb{Q}} \epsilon D - K_X$ is ample.

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- The statement about movable cones follows instead by running a MMP over T and observing that if $D|_{X_0}$ is not movable, then $\mathrm{SBs}(D|_{X_0})$ contains a divisor which must be contracted by this MMP.
- As we have seen above, this divisorial contraction induces a divisorial contraction on X_t for t near 0 and hence $D|_{X_t}$ is not movable.

Mori Chambers

- Recall that if L_i are movable divisors on X such that $\mathrm{Proj} R(L_1) \cong \mathrm{Proj} R(L_2)$, then L_1 and L_2 belong to the same Mori chamber of $\mathrm{Mov}^1(X)$ (we also require that the interior of a Mori chamber is open in $N^1(X)$).

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- The following conjecture is a consequence of the conjecture on nef values and hence holds for several classes of varieties (eg. toric varieties; $\dim X_0 \leq 3$):

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- In particular the nef cones are locally constant.

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- When X_0 is a toric variety, this is in fact a consequence of the following:

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- Note that if X_0 is smooth, then by a result of Bien and Brion, $H^1(T_{X_0}) = 0$.
- In general, we use the fact that Fano varieties are Mori Dream spaces i.e. $h^0(\Omega^1_{X_0}) = 0$ and the Cox ring $C(X_0) = \oplus_{[D]\in Cl(X_0)} H^0(\mathcal{O}_{X_0}(D))$ is finitely generated.

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- Using the extension theorem discussed above, one then shows that sections of $\mathcal{O}_{X_0}(D)$ extend to nearby fibers.
- Since $C(X_0)$ is polynomial, it then follows that the $C(X_t)$ is polynomial and hence that X_t is toric. It is then easy to see that X_0 is rigid.