Quotients by finite equivalence relations

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Introduction

Classical terminology:
"a nodal curve is obtained from a smooth curve by identifying 2 points"

General version:
Y: reduced, non-normal variety/scheme
$\bar{Y}$: normalization

Question: How to reconstruct Y from $\bar{Y}$?

Fiber product: $R := \bar{Y} \times_Y \bar{Y} \subset \bar{Y} \times \bar{Y}$
is an **equivalence relation** on $\bar{Y} \times \bar{Y}$.

Hope: we can recover Y as the "quotient" of $\bar{Y}$ by R.

Set theoretic version: red $R \subset \bar{Y} \times \bar{Y}$;
should work if Y semi-normal

Examples I.

$Y := \big(0 \in L_1 + L_2 + L_3\big) \subset \mathbb{A}^2$,
3 lines through the origin

$\bar{Y} = (0_1 \in L_1) \amalg (0_2 \in L_2) \amalg (0_3 \in L_3)$
3 disjoint lines

R: diagonal and $(0_i, 0_j) : \ 1 \leq i, j \leq 3$
(with reduced scheme structure)

$\bar{Y}/R = \big(0 \in L_1' + L_2' + L_3'\big) \subset \mathbb{A}^3$
3 lines through the origin

Corollary: $Y \neq \bar{Y}/R$.

Question: $Y \mapsto \bar{Y}/\big(\bar{Y} \times_Y \bar{Y}\big)$
is some closure operation.
What is it?

Equivalence relations

Defn. $R \subset X \times X$ equivalence rel. iff:

(1) (reflexive) contains the diagonal Δ_X.

(2) (symmetric) invariant under interchange of factors

(3) (transitive) $\exists$ factorization

$$R_{12} \times_{X_2} R_{23} \to R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3,$$

where $X_i := X$, $R_{ij} := R \subset X_i \times X_j$.

Equivalent: for every scheme T, we get a (set theoretic) equivalence relation

$$\mathrm{Mor}(T, R) \subset \mathrm{Mor}(T, X) \times \mathrm{Mor}(T, X).$$

Finite equivalence relation: The projections $\sigma_1, \sigma_2 : R \rightrightarrows X$ are finite.

Conjecture

$R \subset X \times X$ a finite equivalence relation.
Then $R = X \times_Y X$ for some
finite surjection $f : X \to Y$.

Algebraic translation:
$\mathbf{x} = (x_1, \ldots, x_n)$ etc.
$I := I(\mathbf{x}, \mathbf{y}) \subset k[\mathbf{x}, \mathbf{y}]$ is equiv. rel. iff:
(reflexive): $I \subset (x_1 - y_1, \ldots, x_n - y_n)$.
(symmetric): $f(\mathbf{x}, \mathbf{y}) \in I$ iff $f(\mathbf{y}, \mathbf{x}) \in I$.
(transitive): in $k[\mathbf{x}, \mathbf{y}, \mathbf{z}]$:

$$I(\mathbf{x}, \mathbf{z}) \subset \Big(I(\mathbf{x}, \mathbf{y}), I(\mathbf{y}, \mathbf{z}) \Big).$$

(finiteness): $k[\mathbf{x}, \mathbf{y}]/I$ finite over $k[\mathbf{x}]$.

Conjecture: I generated by elements of the form $f(\mathbf{x}) - f(\mathbf{y})$.

Eisenbud *et alii*: Macaulay program:
– starts with $J \subset k[\mathbf{x}, \mathbf{y}]$
– generates equiv. rel. $I \subset k[\mathbf{x}, \mathbf{y}]$
– checks conjecture.

Examples II.

$X := \mathbb{C}^2$ with $\mathbb{Z}/2$-action $(x, y) \mapsto (-x, -y)$.
$R \subset \mathbb{C}^2 \times \mathbb{C}^2$: graph of action

$$(x_1 - x_2, y_1 - y_2) \cap (x_1 + x_2, y_1 + y_2) =$$
$$(x_1^2 - x_2^2, y_1^2 - y_2^2, x_1y_1 - x_2y_2, x_1y_2 - x_2y_1)$$

Claim: NOT an equivalence relation.
Transitivity: Ideal of $R_{12} \times_{X_2} R_{23}$ is
$$(x_1^2 - x_2^2, y_1^2 - y_2^2, x_1y_1 - x_2y_2, x_1y_2 - x_2y_1,$$
$$x_2^2 - x_3^2, y_2^2 - y_3^2, x_2y_2 - x_3y_3, x_2y_3 - x_3y_2)$$
contains: $x_1^2 - x_3^2, y_1^2 - y_3^2, x_1y_1 - x_3y_3$
does not contain: $x_1y_3 - x_3y_1$.

Remedy: $R^* \subset X \times X$ defined by
$$(x_1^2 - x_2^2, y_1^2 - y_2^2, x_1y_1 - x_2y_2) \subset \mathbb{C}[x_1, y_1, x_2, y_2].$$
Check: R^* is an equivalence relation.
Difference: 1 embedded point at the origin.

Set theoretic equiv. rels.

Transitivity condition

$$R_{12} \times_{X_2} R_{23} \to R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3.$$

replaced by:

$$\operatorname{red}\bigl(R_{12} \times_{X_2} R_{23}\bigr) \to R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3.$$

Geometric quotient: X/R

(1) $q : X \to X/R$ is finite,
(2) for every $\operatorname{Spec} K \to S$, the fibers of
$q : X(K) \to X(K)/R(K)$ are the
$R(K)$-equivalence classes of $X(K)$, and
(3) $q : X \to X/R$ is biggest such.

Question: $R \subset X \times X$, finite
set theoretic equiv. rel.
Is there a geometric quotient X/R?

Note: Finiteness essential, e.g.
$C \subset \mathbb{P}^2$ can not be contracted.

Examples III.

(Holmann, '63)

$X := \operatorname{Spec} k[x_1, y_1^2, y_1^3] \amalg \operatorname{Spec} k[x_2, y_2^2, y_2^3]$

(two copies of $\mathbb{A}^1 \times$ (cuspidal cubic))

R : diagonal and the graph of the shear $(x_2, y_2) \mapsto (x_1 + y_1, y_1)$.

What is $k[u, v^2, v^3] \cap k[u+v, v^2, v^3]$?

$k[u, v^2, v^3] =$
$\left\{ f_0(u) + \sum_{i \geq 2} v^i f_i(u) \ : \ f_i \in k[u] \right\}$ and
$k[u+v, v^2, v^3] =$
$\left\{ f_0(u) + v f_0'(u) + \sum_{i \geq 2} v^i f_i(u) \ : \ f_i \in k[u] \right\}$.

Cor:

i) If char $k = 0$, the intersection is:
$k + \sum_{n \geq 2} v^n k[u]$: non-Noetherian
(need all $v^2 u^m, v^3 u^m$)

ii) If char $k = p$, the intersection is:
$k[u^p] + \sum_{n \geq 2} v^n k[u]$: finitely generated
(need $v^2 u^m, v^3 u^m$ for $m < p$)

Positive characteristic

Thm. S: Noetherian $\mathbb{F}_p$-scheme,
X: essentially of finite type over S
R: finite, set theoretic equivalence rel.
$\Rightarrow$ the geometric quotient X/R exists.

Example (Nagarajan, '68)
char k=2
$K = k(x_1, y_1, x_2, y_2, \dots)$
$R := K[[u, v]]$ (regular of dim. 2)
derivation D_K from K to R:
$\sum_i v(x_{i+1}u + y_{i+1}v)\frac{\partial}{\partial x_i} + u(x_{i+1}u + y_{i+1}v)\frac{\partial}{\partial y_i}$
extended: $D_R|_K = D_K$, $D_R(u) = D_R(v) = 0$
$\sigma : r \mapsto r + D_R(r)$, involution.

Claim: Invariant ring R^σ not Noetherian.

Check: $I_m := (x_1 u + y_1 v, \dots, x_m u + y_m v)$
is strictly increasing sequence.

Easy cases

Prop. 1 Assume $\sigma_i : R \to X$ are flat.
Then $x \mapsto \left[\sigma_{2*}\big(\sigma_1^{-1}(x)\big)\right] \in \mathrm{Hilb}(X)$
gives X/R.

Prop. 2 Assume X normal, R equidim.
Then $x \mapsto \left[\sigma_{2*}\big(\sigma_1^{-1}(x)\big)\right] \in \mathrm{Chow}(X)$
gives X/R.

Prop. 3 Assume $g : X \to S$ is finite and
$R \subset X \times_S X \subset X \times X$. Then

$$X/R = \mathrm{Spec}_S \ker\left[g_*\mathcal{O}_X \stackrel{\sigma_1^*-\sigma_2^*}{\longrightarrow} g_*\mathcal{O}_R\right].$$

(Note: "counter example" in literature)

Inductive plan

Step 1. $X^n \to X$ normalization, $d := \dim X$

R^n: pull back of R to X^n

$X^{nd} \subset X^n$, $R^{nd} \subset R^n$: d-dim. parts

X^{nd}/R^{nd} exists by Prop 2.

Step 2. $Z \subset X^n$: where the lower dim. parts of R^n live.

Z_1: image of Z in X^{nd}/R^{nd}

$R^n|_Z$ descends to an equiv. rel R_{Z_1} on Z_1.

Step 3. **Assumption:** Z_1/R_{Z_1} exists.

Step 4. Take universal pushout

$$\begin{array}{ccc} Z_1 & \hookrightarrow & X^n/R^{nd} \\ \downarrow & & \downarrow \\ Z_1/\big(R^n|_Z\big) & \hookrightarrow & X^n/R^n. \end{array}$$

Step 5. Define X^* as the diagonal image:

$$\begin{array}{ccccc} & & X^n & & \\ & \swarrow & \downarrow & \searrow & \\ X^n/R^n & \leftarrow & X^* & \rightarrow & X \end{array}$$

Step 6. $X^* \to X$: finite homeomorphism.

If invertible then Prop 3. gives X/R.

Factoring the Frobenius

Prop. X, Y: finite type over $\mathbb{F}_p$
$g : X \to Y$ finite, universal homeo.
$\Rightarrow$ for $q = p^r \gg 1$ there is

$$F^q : X \xrightarrow{g} Y \xrightarrow{\bar{g}} X^{(q)}.$$

Key poin: If $I \subset A$ nilpotent, then
for $q = p^r \gg 1$, $a \mapsto a^q$ gives
well defined splitting $A/I \to A$.

Cor. In Step 6, $X^* \to X$, is
essentially invertible.

Degeneration of varieties

Δ: disc, $\Delta^* := \Delta \setminus \{0\}$
Given $\pi^* : X^* \to \Delta^*$ with
smooth, general type fibers,
how can one extend it to $\pi : X \to \Delta$?

Step 1. Take any extension $\pi_1 : X_1 \to \Delta$

Step 2. (semi-stable reduction)
Blow up to $\pi_2 : X_2 \to \Delta$ s.t.
$\pi_2^{-1}(0)$ is simple normal crossing

Step 3. (minimal model program)
Canonical model $\pi : X = X_2^{\text{can}} \to \Delta$.

Step 4. Central fiber has
i) semi-log-can. singularities (=**slc**)
ii) ample canonical class
iii) possibly reducible.

What to do if $X^* \to \Delta^*$ has
reducible general fibers?

Failure of MMP

Problem: MMP fails for reducible case:
There is a simple normal crossing surface whose canonical ring is **not** finitely generated.

Solution:

i) Take normalized components $\bar{X}_i^*$.
ii) Construct each $\bar{X}_i \to \Delta$.
iii) Glue the $\bar{X}_i$ together.

Questions:

1. What is special about $\amalg \bar{X}_i \to X$ if the $\bar{X}_i$ are log canonical?
2. How to use this to construct $\amalg \bar{X}_i / R$?

Inductive semi-normality

Observation: If X is semi normal, then $X^* \to X$ in Step 5 is an isomorphism, **but** we need this inductively for Z_1 too.

Step 3. Assumption: Z_1/R_{Z_1} exists.

Step 4. Take universal pushout

$$\begin{array}{ccc} Z_1 & \hookrightarrow & X^n/R^{nd} \\ \downarrow & & \downarrow \\ Z_1/\big(R^n|_Z\big) & \hookrightarrow & X^n/R^n. \end{array}$$

Step 5. Define X^* as the diagonal image:

$$\begin{array}{ccccc} & & X^n & & \\ & \swarrow & \downarrow & \searrow & \\ X^n/R^n & \leftarrow & X^* & \rightarrow & X \end{array}$$

Step 6. $X^* \to X$: finite homeomorphism. If invertible then Prop 3. gives X/R.

Stratifications

(X, S_*): $X = \cup_i S_i X$, $\dim S_i X = i$.
Assume: $\cup_{i \le j} S_i X$ is closed $\forall\, j$.
$\pi : X' \to X$ stratifiable if $\pi^{-1}\big(S_i X\big)$ has pure dimension i. Then get $\big(X', \pi^{-1} S_*\big)$.

Conditions:

(N) (normal strata) each $S_i X$ is normal.

(SN) (seminormal boundary): X and the boundary $BX = \cup_{i < \dim X} S_i X$ are seminormal.

(HN) (hereditarily normal strata)
- (a) X satisfies (N),
- (b) $\pi : \bar{X} \to X$ is stratifiable, and
- (c) $B\big(\bar{X}\big)$ satisfies (HN).

(HSN) (hereditarily seminormal boundary)
- (a) X satisfies (SN),
- (b) $\pi : \bar{X} \to X$ is stratifiable, and
- (c) $B\big(\bar{X}\big)$ satisfies (HSN).

Final steps

Theorem. Assume that
i) (X, S_*) satisfies (HN) and (HSN),
ii) $\sigma_i : R \to X$ are stratified.
$\Rightarrow X/R$ exists.

Theorem. (K-Kovács)
Let (X, D) be log canonical.
Set $S_i X :=$
$\cup$(i-dim lc centers)$\setminus\cup$($<i$-dim lc centers).
$\Rightarrow (X, S_*)$ satisfies (HN) and (HSN).

Ambro, Fujino: satisfies (N) and (SN).

Adjunction conjecture:
Let $Z \subset X$ be an lc center. Then
$\big(K_X + D\big)|_Z = K_Z + D_Z$.

Classical case: $D = Z$, then $D_Z = 0$.

Genereal case:
Still not known, so
we have to go around it.

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Quotients by finite equivalence relations

<u>Example.</u> Can think of a nodal curve as a smooth curve quotiented by a relation identifying two points.

General version: Start with a reduced non-normal variety, Y. Let $\bar{Y}$ be the normalization, and we want to reconstruct Y as the "quotient" of $\bar{Y}$ by an equivalence relation.

Equiv. relation - try fiber product $R := \bar{Y} \times_Y \bar{Y}$.

Set theoretically, looks promising - two points in $\bar{Y}$ are identified if they have the same image in Y.

<u>Example.</u> Y - 3 lines in $\mathbb{A}^2$ through origin. $O \in L_1, L_2, \& L_3$

$\bar{Y}$ - 3 disjoint lines. $O_1 \in L_1$, $O_2 \in L_2$, $O_3 \in L_3$

R - diagonal and (O_i, O_j) $1 \le i, j \le 3$.

$\bar{Y}/R$ = 3 lines through origin in $\underline{\mathbb{A}^3}$.

$Y \neq \bar{Y}/R$ (Y was "more singular" than $\bar{Y}/R$).

$Y \mapsto \bar{Y}/\bar{Y} \times_Y \bar{Y}$ can be thought of as a closure operation. We want to study this.

Def. $R \subset X \times_S X$ is an *equivalence relation* iff:

1) (reflexive) Contains the diagonal Δ_X.

2) (symmetric) fixed under interchange of factors.

3) (transitive) $\exists$ factorization

$$R_{12} \times_{X_2} R_{23} \longrightarrow R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3 .$$

$$X_i = X, \quad R_{ij} = R \subset X_i \times X_j .$$

(note set theoretically, this is what I expect. $a \sim b$, $b \sim c \to a \sim c$ already.

Def. Finite equivalence relation: Projections $\sigma_1, \sigma_2 : R \rightrightarrows X$ finite.

Q. Does every finite equiv. rel. arise as a fiber product?

Conjecture. $R \subset X \times X$ a finite equivalence relation, then $R = X \times_Y X$ for some finite surjection $f : X \to Y$.

Algebraic translation.

$I := I(\underline{x}, \underline{y}) \subset k[\underline{x}, \underline{y}]$ is an equiv. rel. iff

1) reflexive – $I \subset (x_1 - y_1, \dots x_n - y_n)$

2) symmetric – $g(\underline{x}, \underline{y}) \in I$ iff $g(\underline{y}, \underline{x}) \in I$.

3) transitive – $I(\underline{x}, \underline{z}) \subset (I(\underline{x}, \underline{y}), I(\underline{y}, \underline{z}))$.

finiteness – $k[\underline{x}, \underline{y}]/I$ finite over $k[\underline{x}]$.

In this algebraic translation, above conjecture is:

I generated by elements of the form $g(\underline{x}) - g(\underline{y})$.

(ie.

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\sigma_2} & X \sim k[\underline{y}] \\ \sigma_1 \downarrow & & \downarrow f \\ k[\underline{x}] \sim X & \xrightarrow{f} & Y \end{array}$$

In this picture the ideal for $X \times_Y X$ should be generated by $g(\underline{x}) - g(\underline{y})$ for g pull back of a function on Y.)

Example $X := \mathbb{C}^2$ with $\mathbb{Z}/2$ action $(x,y) \to (-x,-y)$.

identify each point with self $- I_1 = (x_1 - x_2, y_1 - y_2)$ and with its neg $-$ $I_2 = (x_1 + x_2, y_1 + y_2)$. Let $R = I_1 \cap I_2$.

Not an equivalence relation b/c transitivity fails.

But if instead take $R^* \subset X \times X$ given by $(x_1^2 - x_2^2, y_1^2 - y_2^2, x_1 y_1 - x_2 y_2)$, this is an equiv. rel. Note the difference between R and R^* is R^* contains an embedded point at the origin.

Scheme structure in last example made things more complicated, So let us try studying set theoretic equiv. rel.

replace transitivity condition with

$$\mathrm{red}(R_{12} \times_{X_2} R_{23}) \longrightarrow R_{13} \longrightarrow X_1 \times_S X_3 .$$

Now define geometric quotient by an equiv. rel.
(essentially "biggest" scheme you can have that has all the required points identified.)

Q. Given $R \subset X \times X$ finite set theoretic equiv. rel., is there a geometric quotient X/R?

Example. (usually answer to Q is no.).
(Holmann)

$\mathrm{Spec}\,(k[x_1 y_1^2 y_1^3]) \amalg \mathrm{spec}\,(k[x_2 y_2^2 y_2^3])$

X = two copies of ($\mathbb{A}^1$ x cuspidal cubic).

R = diagonal and graph of $(x_2 y_2) \to (x_1 + y_1, y_1)$.

char. 0:

The ring of functions for the thing that would be the geometric quotient ($k[u, v^2, v^3] \cap k[u+v, v^2, v^3]$) is not even notherian. So, geom. quotient does not exist. (For set theoretic equiv. rel.)

char p:

Actually, in this case, geom. quotient does exist.

(Geom. quot. does exist for scheme theoretic equiv rel.: in this example if you try to make a scheme theoretic equiv. rel., have to take transitive closure ∞'ly many times. (this is where the "non-notherian" shows up.).

Thm. In char. $p > 0$, for set theoretic equiv. rels, geom. quotient always exists.

In char 0, for certain cases easy to show geom. quot does exist.

Prop 1. $\sigma_i : R \to X$ are flat $\to$ $X/_R$ exists

Prop 2. X normal, R equidim $\to$ $X/_R$ exists.

(eg. node not equidim, b/c R contains diag, dim 1, & $p \neq q$, dim 0.)

Prop. 3. $g : X \to S$ finite and $R \subset X \times_S X \subset X \times X$ $\longrightarrow$ $X/_R$ exists.

Inductive plan to construct geom. quotients, in cases beyond the prev. 3 "easy" cases.

Idea: Use prop 2 - want X normal & R equidim.

Well, given arbitrary X, R, not the case, but can take normalization $X^n \to X$, pull back R to R^n on X^n; and first deal w/ top dimen. part of R^n, & with the lower dim part inductively.

Step 2.

⋮

Step 6. $X^* \to X$ finite homeomorphism, and <u>if invertible</u> then Prop 3 gives X/R.

(Note. In Holmann example, this is exactly where the problem in char. 0 arises.)

In char $p > 0$, step 6 is ok, b/c by factoring the Frobenius map, desired map is invertible, so the plan goes through.

<u>Degeneration of varieties</u> (the original intended application of this geometric quotient work.)

Set up: Δ disc, Δ^* punctured disc.

Given $\pi^* : X^* \to \Delta^*$ with smooth, gen. type fibers, how can it be extended across the puncture to $\pi : X \to \Delta$?

$\exists$ good solution in this case (i.e. smooth fibers away from puncture.)

Step 1. Fill in puncture any way.

Step 2. Blow up the new fiber until $\pi^{-1}(0)$ is simple normal crossing.

Step 3. Run minimal model program on the variety X we've constructed.

Step 4. We've succeeded in "filling in the puncture" with a central fiber which:

i) has semi log canonical sings.

ii) has ample canonical class

iii) is possibly reducible.

We hope that we can find a space closed under degenerations though, it is unpleasant that we've added new kind of variety when we started with smooth ones.

But we hope to study this new class of varieties instead, want it to be closed under degeneration.

So, Q: How to degenerate $X^* \to \Delta^*$ if we start with general fibers which are reducible?

We want to do exactly what we did before, but problem: Min. model. program fails if we try to do this directly - problem with reducible variety.

($\exists$ simple normal crossing (transversal self intersection) surface (in partic. reducible) in $\mathbb{P}^n$ whose canonical ring is not fin. gen.)

Instead, propose:

i) Break X^* into irreducible components X_i^* normalize individually, $\bar{X}_i^*$.

ii) Construct each $\bar{X}_i \to \Delta$

iii) Use fin. equiv rel. to glue the $\bar{X}_i$ together.

Hope that because $\bar{X}_i$ are now log canonical (instead of just semi log can.) we are able to construct quotient by equiv. rel. In general can't construct these quotients, but hope able to in this special case.

We try this, and find:

Good! So the construction goes through!

The map in step 6 of inductive construction of X/R is an iso for semi normal variety. But because have inductive construction, need to somehow ensure that the smaller spaces that arise during the induction are also seminormal. We want to use fact that $\bar{X}_i$ are log canonical to do this.

ie. need inductive semi normality.

Solution: Use stratification.

Write (X, S_*) : $X = \cup_i S_i X \quad \dim S_i X = i$.

Impose several normality / seminormality conditions on the strata.

Theorem A: Assume that (X, S_*) is stratification satisfying normality conditions, and $\sigma_i : R \to X$ stratified morphisms. (Preimage of i dim'l stratum of X under σ_i must lie in i dim'l piece of R.) Then X/R exists.

(ie. under these conditions, all steps in proposed construction of X/R can be carried out.)

Theorem B: (Kollár - Kovács).

Let (X, D) be log canonical.

Define stratification $S_i X := \bigcup(i\text{-dim log canonical centers}) \setminus \bigcup(<i \text{ dim log canonical centers.})$

$\Rightarrow (X, S_*)$ satisfies the normality conditions of thm. A.

Point: We wanted to construct the central fiber (in the degeneration of reducible varieties problem) by constructing it in pieces and then quotienting by an equiv. rel. to glue the pieces together correctly. In that application, the central fiber was log canonical, so thm B says we can use that property to put a stratification on the central fiber satisfying the conditions required in thm. A, which says we can in fact construct the geometric quotient. This geometric quotient results in a variety that is a degeneration of the family we started with. Thus we suceed in finding a class of varieties closed under degeneration and in giving a recipie for constructing the central fiber in all cases.