The Hodge theory of character varieties

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Contents

1 The decomposition theorem (DT)

dt

- $S^1 \to S^3 \xrightarrow{f} S^2$ (Hopf bundle); Leray spectral sequence (LSS) is not E_2-degenerate:

$$d_2 : H^0(S^2, \mathbb{Z}) \xrightarrow{\simeq} H^2(S^2, \mathbb{Z}), \qquad 1 \longmapsto \text{Euler class.}$$

- Deligne ('68): E_2-degeneration for $X \xrightarrow{f} Y$ smooth, proper of q.proj. var:

$$R^i := R^i f_* \mathbb{Q},\ R^i_y = H^i(X_y, \mathbb{Q}) : \quad Rf_*\mathbb{Q} \simeq \bigoplus_i R^i[-i],$$

  $Rf_*\mathbb{Q}$" = "$\{R^i\}$ + {trivial extension data}; Leray filtration "visible:" $\bigoplus_{\leq i}$.

- Goresky-MacPherson ('70s): Y alg.var. has intersection cohomology groups $IH^*(Y)$:
  – Y smooth : $IH = H$; Poincaré duality: $IH^{n+j}(Y) \times IH^{n-j}_c(Y) \longrightarrow \mathbb{Q}$ perfect.
  – IC_Y complex of sheaves on Y: $IH(Y) = \mathbb{H}(Y, IC_Y)$ **(vague with indices)**.
  – Y smooth: $IC_Y = \mathbb{Q}_Y$; $IC_Y(L)$: twisted version, L local system on $Y^0 \subseteq Y_{reg}$.

- B-B-D-G (1982) **DT**: $X \xrightarrow{f} Y$ proper of algebraic varieties. Then

$$Rf_* IC_X \simeq \bigoplus_b IC_{Z_b}(L_b)[d_b]$$

  for a *uniquely determined collection* $b \in B$ of
  $Z_b \subseteq Y$ closed irreducible, $0 \neq L_b$ local system on $Z_b^0 \subseteq Z_b$, and $d_b \in \mathbb{Z}$.
  – DT for f proper and smooth is Deligne '68;
  – B-B-D-G used l-adic sheaves on varieties over finite fields;
  – M. Saito ('80's): vast generalization that incorporates Hodge theory.

- If X is smooth, the lhs does not get simpler: hence the fundamental nature of IC_Y. Given f_U, IC_Y recipe governs Rf_*.

  If $f : X \to Y \leftarrow T : g$ share L on U, share $IC_Z(L)$ on Y: g simpler, helps f.

- Examples of DT:
  1. resolution of surfaces: $Rf_*\mathbb{Q} \simeq IC_Y \oplus \mathbb{Q}_v^{\#-1}[-2]$;
  2. surface to curve: $Rf_*\mathbb{Q} \simeq \mathbb{Q}_Y \oplus IC_Y(R^1) \oplus \mathbb{Q}_v^{\#-1}[-2] \oplus \mathbb{Q}[-2]$;
     implies LICT: $H^1(X_o) \stackrel{epi}{\to} H^1(X_t)^{\pi_1} \subseteq H^1(X_t)$.

- The determination of the **supports** Z_b is an important and difficult problem.

  The Z_b are among the closed irreducible subvarieties $Z \subseteq Y$ s.t.

  1. $\exists\, Z^0 \subseteq Z$ over which all $R^i f_* \mathbb{Q}$ are local systems, and
  2. Z is maximal with this property.

  Still, given such a Z, in general it is hard to check if it supports a summand in DT.

2 Ngô support theorem

st

- B.C. Ngô has proved the fundamental lemma in the Langlands program (omit).

  One crucial result he proves is geometric and deals with the determination of the Z_b for the *Hitchin map* $h : M \to \mathbb{A}$ (to fix idead: $G = GL_n$).

- X a cpt Riemann sfce.

  M is the moduli space of stable rank n Higgs bundles (E, ϕ) on X, $\phi : E \to EK_X$.

  h sends ϕ to its characteristic polynomial (viewed as a section of a bundle on X).

  h is proper, flat of some rel.dim. $= d$.

  $U \subseteq \mathbb{A}$ is a certain big open set over which h has **reduced** fibers.

  Take: $f := h_U : M_U \to U$.

Theorem 2.1 (Ngô, '06)
$Z \subseteq U$ closed irreducible appears in DT for f
IFF
$\exists\, Z^0 \subseteq Z$ (open dense) s.t. $R^{2d} f_* \mathbb{Q}_{|Z^0}$ loc.const. and Z is maximal with this property.

Why is this remarkable?:

All the supports Z_b are determined exactly by $R^{2d} f_* \mathbb{Q}$ only!
$R^{2d} f_* \mathbb{Q}$ is particularly understandable: it is the sheaf of irreducible components of the fibers of f; the monodromies involved with $R^{2d} f_* \mathbb{Q}$ factor through finite groups.

2.1 Some of the ideas in proof of the support theorem

Abelian variety A acting with finite stabilizers on a T. $H_*(A)$ homology algebra:

$H_c^*(T)$ free gr.-$H_*(A)$-module.

- $f: M_U \to U$, proper, flat, rel.dim d. $U \ni u \longmapsto \{X_u \to X\}$ spectral curve.

M_u moduli of torsion-free rk 1 sheaves of degree 1 on X_u.
$Pic(X_u)$ acts on M_u. $Pic(X_u) \supseteq P_u$ =Id-component. Chevalley devissage:

$$1 \to R_u^{\delta_u} \to P_u^d \to A_u^{d-\delta_u} \to 1, \quad A_u \,\text{Abelian var.},\ R_u \ \text{affine var.}$$

- Z_b occurs in DT as support of summands of $Rf_*\mathbb{Q}_M$. We group them and get

$$\text{a graded object} \qquad \bigoplus_{n \in \text{Occ}_b} IC_{Z_b}(L_b^n)[\ldots],$$

$$\text{Occ}_b \subseteq [-d,d]: \quad \overset{-d}{-} \ \ldots \overset{n_-}{*} \ \ldots * \ldots \overset{0}{-} \ \ldots * \ldots \overset{n^+}{*} \ \ldots \overset{d}{-}$$

and the symmetry about 0 is due to Poincaré-Verdier duality. Clearly:

$$n^+ := \max n = \frac{1}{2}(n^+ - n_-).$$

- Work on $U^0 \subseteq U$, where $IC_{Z_b}(L_b)$ is L_b, so that

$$\text{DT} \Longrightarrow L_b^{n^+} \text{ summand of } R^{\#} f_* \mathbb{Q}.$$

GOAL: *show that $\# \geq 2d$.* Then (dim fibers $= d$) $\Longrightarrow$ ($\# = 2d$) and:
$L_b^{n^+}$ direct summand of $R^{2d} f_* \mathbb{Q}$, the support theorem follows.

STRATEGY: ($Z_b \ni z$ general)

$$\# = 2d + (\text{codim}_S Z_b - \delta_z)^{\overset{?}{\geq} 0} + [n^+ - (d - \delta_z)]^{\overset{?}{\geq} 0} \overset{?}{\geq} 2d.$$

- The first inequality: deformation theory of Higgs bundles and RR on X.
- The second inequality.

The graded $L := \oplus_{n_-}^{n^+} L_b^n[\ldots]$ has length $l(L) = 2n^+$.
2^{nd} inequality becomes: $l(L)(=2n^+) \geq 2(d - \delta_z)$.
Since $l(H_*(A_z)) = 2 \dim A_z = 2(d - \delta_z)$, we would like to prove:

L_z free gr.-$H_*(A_z)$-module.

P_z acts on M_z with affine stabilizers.
Assume $\exists$ splitting $A_z \to P_z$. Ok over finite field and that's enough.
$\Longrightarrow$ A_z acts on M_z with finite stabilizers and

$H^*(M_z)$ free gr.-$H_*(A_z)$-module containing L_z as a graded vector subspace.

Delicate technical point: L_z can be exhibited as a free gr-$H_*(A_z)$-mod.

We are done: $\quad 2n^+ = l(L) \overset{\text{freeness}}{\geq} l(H_*(A_z)) = 2(d - \delta_z)$, as wanted.

3 The perverse filtration and the Lefschetz hyperplane thm

For simplicity, I use the the language of filtrations, instead of the one of spectral sequences. Also, I omit indices.

- Algebraic topology.

  $\pi : E \to B$ a topological fiber bundle.

  Leray filtration on $H^*(E, \mathbb{Z}) \supseteq \mathrm{Im}\,\{H^*(Y, \tau_{\leq \bullet} Rf_* \mathbb{Z})\}$.

  B_* an n-flag of closed subspaces (scheleta): $B_0 \subseteq \ldots \subseteq B_n = B$, e.g. CW-complexes.

  $E_* := \pi^{-1}(B_*)$ the preimage flag.

  Flag filtration on $H^*(E) \supseteq \mathrm{Ker}\,\{H^*(E) \to H^*(E_\bullet)\}$.

  In general, Leray and flag filtrations are **unrelated**.

  If π-cellularity of B_*: $H^*(Y_p, Y_{p-1}, R^q\pi_*\mathbb{Z}) = 0$, $* \neq p$ (e.g. Y_* cell cplex), then

  Leray filtration = E_*-flag filtration.

- Algebraic geometry: anything similar with the scheleta algebraic subvarieties?

  Arapura ('05): Let $X \xrightarrow{f} Y$ be proj. of q.proj.var. (For simplicity: $Y^n \subseteq \mathbb{A}^N$.)

  Then there exists an n-flag Y_* of closed algebraic subvarieties of Y such that Leray filtration = X_*-flag filtration (here, $X_* = f^{-1}Y_*$ on $H(X)$).

  Y_*: c.i. of Y with high degree hypersurfaces in special position.

  Corollary (M. Saito (80's)): *the subspaces of the Leray filtration are mHss.*

- In addition to the Leray filtration on $H^*(X)$, there is the *perverse Leray filtration*:

  Leray: $\mathrm{Im}\, H(Y, \tau_{\leq *} Rf_*\mathbb{Z}) \subseteq H(X, \mathbb{Z})$, p-Leray: $\mathrm{Im}\, H(Y, {}^{\mathfrak{p}}\tau_{\leq *} Rf_*\mathbb{Z}) \subseteq H(X, \mathbb{Z})$.

  Replace $\mathbb{Z}_X$ with any constructible complex C on X, and also H^* with H^*_c (w/caution).

  Theorem 3.1 (d-Migliorini, '08) *Let Y be quasi projective and $X \xrightarrow{f} Y$ be any algebraic map. (For simplicity, $Y^n \subseteq \mathbb{A}^N$).*

  Then there is an n-flag Y_* on Y such that, on $H^*(X, C)$ and on $H^*_c(X, C)$, p-Leray filtration = flag filtration for $X_* = f^{-1}Y_*$

  Y_*: c.i. of Y with linear sections in general position.

  Several applications to Hodge theory can be given.

  Why LHT?: The Lefschetz hyperplane theorem for perverse sheaves is used as a vanishing theorem to ensure the f-cellularity-type condition

  $$H^*(Y_p, Y_{p-1}, {}^{\mathfrak{p}}R^q f_* C) = 0 \ \ \forall * \neq 0.$$

4 Character varieties

Joint work in progress with T. Hausel and L. Migliorini.

- X compact Riemann surface.

  M_D moduli sp. of Higgs bundles of degree 1 (of rk 2 to fix ideas). Smooth q.proj.

  M_B the character variety. Smooth affine. Parametrizes equivalence classes of

  $$\pi_1(X - pt) \to GL_2(\mathbb{C}), \qquad \gamma_{pt} \longrightarrow -\mathrm{Id}.$$

  Non Abelian Hodge theorem: $\theta : M_D \overset{\text{diffeo}}{\simeq} M_B$ They are not biholomorphic.

  Hausel-Villegas ('06): *mHs of $H^*(M_B)$: there are tautological classes:*

  $$\alpha \in H^2, \quad \{\psi_i\}_{i=1}^{2g(X)} \in H^3, \quad \beta \in H^4$$

  which are ring generators, have weight 4 and products have the expected weight.

  Curious hard Lefschetz (Hausel-Villegas '06): $\alpha^i : Gr_W H(M_B) \simeq Gr_W H(M_B)$.

  It is curious because M_B is affine, and $\alpha \in H^2$ has weight 4.

  It is curious also because it is there for no apparent reason.

  Note that: θ does not preserve mHS. In fact, $H^*(M_B)$ mixed, but $H^*(M_D)$ is pure.

- **Conjecture (d-Hausel-Migliorini):**

  $\theta^* : H^*(M_B) \to H^*(M_D)$ tranforms W filtr. into p-Leray filtr. (for $h : M_D \to \mathbb{A}$) and CHL into the RHL ($\alpha^i : {}^{\mathfrak{p}}R^{-i}h_*\mathbb{Q}[\dim M_D] \simeq {}^{\mathfrak{p}}R^i h_*\mathbb{Q}[\dim M_D]$).

  OK in low genus, and for the "right" reasons.

  In general, we need to verify that all products $\alpha^\# \psi_i^\# \beta^\#$ are in the right place in pLf.

  By d-M ('08): conjecture ok if products vanish over general linear subspaces of right dimension.

  $\alpha^{k \leq rel.dim.}$ should never be zero: OK since α is h-ample.

  ψ_j should vanish over a point: OK by Thaddeus.

  β should vanish over a general line: we prove this **using Ngô support theorem**.

  Other products: similar but more involved analysis involving degenerate spectral curves (in progress).

- As to β we prove that it is zero over the preimage of a generic line by using Ngô support theorem and the fact that β is a multiple of $c_2(M)$.

MARK DE CATALDO: THE HODGE THEORY OF CHARACTER VARIETIES

NOTES BY KARTIK VENKATRAM

There will be four parts to this talk:
(1) Decomposition theorem
(2) Ngo's support theorem
(3) The perverse filtration and Lefschetz theorem
(4) Character varieties

1. Decomposition Theorem

Consider the Hopf fibration $S^1 \to S^3 \to S^2$: the associated Leray spectral sequence has a nondegenerate map

$$d_2 : H^0(S^2, \mathbb{Z}) \to H^2(S^2, \mathbb{Z}), 1 \mapsto e \tag{1}$$

on the E_2 page, where e is the Euler class.

Theorem 1 (Deligne '68). *Let $f : X \to Y$ be a proper smooth map of quasi-projective varieties. Then the Leray spectral sequence always degenerates: in particular, defining the sheaves R^i by $R^i_y = H^i(F_y)$, we have a decomposition $Rf_*\underline{\mathbb{Q}}_X = R^0 \oplus R^1[-1] \oplus R^2[-2] \oplus \cdots$.*

For Y an algebraic variety, Goresky-MacPherson defined *intersection cohomology* groups $IH^*(Y)$, which are isomorphic to ordinary cohomology groups when Y is smooth and possess Poincaré duality in general. Moreover, there is a complex of sheaves IC_Y on Y s.t. $H^*(IC_Y) = IH^*(Y)$. If Y is smooth, $IC_Y \cong \underline{\mathbb{Q}}_Y$. For the decomposition theorem, we need to consider a twisted variant $IC_Y(L)$ of this construction, where L is a local system on $Y^0 \subset Y^{reg} \subset Y$.

Theorem 2 (BBDG). *For $f : X \to Y$ a proper map, $Rf_*IC_X \cong \oplus IC_{Z_b}(L_b)[d_b]$, where $b \in B$ for B finite, Z_b an irreducible subvariety of Y, L_b a local system on Z_b, $d_b \in \mathbb{Z}$.*

Even if X is smooth, this given an interesting decomposition of $Rf_*\underline{\mathbb{Q}}_X$ as intersection cohomology sheaves. The original proof of this theorem uses ℓ-adic cohomology, so does not yield Hodge-theoretic results. M. Saito gave a complete generalization to the complex case, including the case of mixed Hodge structures.

Moral example: let $f : X \to Y, g : T \to Y$ be maps from completely different varieties. The intersection cohomology of X still decomposes according to the same local systems on Y.

Example. Consider a map $f : X \to Y$ between surfaces with an with an extra component over $\sigma \in Y$. Then $Rf_*\underline{\mathbb{Q}}_X \cong IC_Y \oplus \underline{\mathbb{Q}}_\sigma^{\#-1}[-2]$.

Example. Let $f : X \to Y$ be a map from a surface to a curve, with extra components over $p, q \in Y$. Then $Rf_*\underline{\mathbb{Q}}_X \cong \underline{\mathbb{Q}}_Y \oplus IC_Y(R^1) \oplus \underline{\mathbb{Q}}_p[-2] \oplus \underline{\mathbb{Q}}_q[-2]$. This implies the invariant cycle theorem, i.e. that $H^1(X_\sigma) \to H^1(X_\eta) \subset H^1(X_\eta)$.

2. Ngo's Support Theorem

A crucial part of the proof of this theorem is geometric and deals with the precise determination of the Z_b's appearing in the decomposition theorem for the map $h : M_D \to \mathbb{A}$, where M_D is the moduli space of stable Higgs bundles (E, ϕ), $\mathbb{A}$ is affine space, and h is .... h is proper and flat of relatiive dimension d. For $U \subset \mathbb{A}$, we have

Theorem 3 (Ngo)**.** *An irreducible subvariety $Z \subset U$ appears in the decomposition $\Leftrightarrow \exists Z^0 \subset Z$ s.t. $R^{2d}f_*\underline{\mathbb{Q}}|_{Z^0}$ is locally constant and Z is maximal.*

Note that $R^{2d}f_*\underline{\mathbb{Q}}|_{Z^0}$ has finite monodromy because it is the linearization of a sheaf of finite sets.

3. Perverse Filtration and Lefschetz

Let $\pi : E \to B$ be a fibration: then we have a Leray filtration $H^*(E, \mathbb{Z}) \supset \Im(H^*(B, \tau_{\leq 0}Rf_*\mathbb{Z}))$. In algebraic topology, we can read this by taking a filtration by skeleta on B, i.e. an m-flag of closed subspaces $B_0 \subset B_1 \subset \cdots \subset B_n = B$, and pulling it back to $E_* = f^{-1}B_*$ on E. The date of the flag gives you a filtration on cohomology, called the *flag filtration* Ker $HE \to HE_*$. Choosing the flag well, we find that the Leray filtration will be the same: specifically, we want to choose B_* so that $H^j(B_p, B_{p-1}, R^q\pi_*\mathbb{Z}) = 0$ for all $j \neq p$ and all q. We say that such a B_* is π-cellular.

This idea was adapted to algebraic varieties by Arapura in 2005: given a projective morphism $f : X \to Y$ (arbitrary singularities) of quasi-projective varieties (assume $Y^m \subset \mathbb{A}^N$ for simplicity), then $\exists$ a flag of algebraic subvarieties $Y_* \subset Y$ s.t. $X_* = f^{-1}Y_*$ is the Leray filtration.

Corollary 1 (M. Saito)**.** *Under these assumptions, the Leray subspaces of $H(X)$ are mixed Hodge substructures.*

Consider the spaces $H(Y, {}^p\tau_{\leq 0}Rf_*\underline{\mathbb{Z}}) \subset H(X, \mathbb{Z})$, where ${}^p\tau_{\leq 0}$ denotes the perverse filtration. Take any $C \in D_X$, a certain full subcategory of the derived category of sheaves on X: then for both H and H_c we have

Theorem 4. *For Y quasi-projective, $f : X \to Y$ is any morphism, $\exists Y_* \subset Y$ s.t. the associated X_* is the perverse Leray filtration.*

Lefschetz appears when you want to translate the condition of π-cellularity to this situation: we want our flag to satisfying the vanishing condition

$$H^k(Y, Y_{p-1}, {}^p R^q f_* C) = 0 \forall k \neq 0, q \tag{2}$$

The previous theorem is then a consequence of homological algebra.

4. Character Varieties

(Joint work in progress with Hausel and Migliorini) Let X be a curve of genus g, $h : M_D \to \mathbb{A}$ smooth, quasi-projective, not affine. Let M_B parameterize $\pi_1(X \setminus \{\text{pt}\}) \to GL_2(\mathbb{C}), \gamma \to -Ic$. Then M_B is affine smooth, and its fibers are generically abelian varieties. Hausel and Villegas make explicit the mixed Hodge structures of $H^*(M_B)$. Consider the multiplicative generators of the cohomology ring $\alpha \in H^2, \beta \in H^4, \psi_i \in H^3$ for $i = 1, \dots, 2g$ (all have weight 4). They prove a *curious Hard Lefscetz* $\alpha^i : \mathrm{Gr}_W HM_B \to \mathrm{Gr}_W HM_B$, where W comes from the α, β, ψ_i.

Conjecture 1. *$\theta^* : HM_B \xrightarrow{\sim} HM_D$ exchanges the weight filtration with the perverse Leray filtration, and the curious Hard Lefschetz with the relative Hard Lefschetz $\alpha^i : {}^p R^{-i} f_* \to {}^p R^i f_*$.*

27 January 2009

Speaker: Mark de Cataldo

Title of the talk: The Hodge theory of character varieties.

There will be 4 parts to this talk:

(1) Decomposition theorem

(2) Ngô support theorem

(3) The perverse filtration and Lefschetz theorem

(4) Character varieties

(1) Let's begin with a topological example. Consider the Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$

The Leray spectral sequence is nondegenerate at E_2.

$$d_2 : H^0(S^2, \mathbb{Z}) \longrightarrow H^2(S^2, \mathbb{Z})$$
$$1 \longmapsto [\text{Euler class}]$$

However, in the algebraic setting we have

<u>Theorem</u> (Deligne 1968)

If $X \xrightarrow{f} Y$ is a proper and smooth morphism of quasi-projective varieties then the Leray spectral sequence always degenerates. In fact, if R^i are the sheaves with $R^i_y = H^i(F_y)$ where F_y is the fiber over y, then

$$Rf_* \underline{\mathbb{Q}}_X \simeq \bigoplus_{q \geq 0} R^q[-q] \quad \text{(in the derived category)}$$

Note that the above decomposition itself implies that the Leray spectral sequence degenerates.

In 1970s Goresky - MacPherson introduced intersection cohomology $IH^*(Y)$ for algebraic varieties Y. If Y is smooth $IH^*(Y)$ is isomorphic to the ordinary cohomology $H^*(Y)$. In general, $IH^*(Y)$ possesses Poincaré duality.

Moreover, there is a complex of sheaves IC_Y on Y such that $H^*(IC_Y) = IH^*(Y)$. If Y is smooth, $IC_Y \simeq \underline{\mathbb{Q}}_Y$. For the decomposition theorem we need to consider the twisted variant $IC_Y(L)$ of this construction, where L is a local system on $\underset{\text{open dense}}{Y^0} \subset Y^{reg} \subset Y$

<u>Theorem</u> (<u>B</u>eilinson - <u>B</u>ernstein - <u>D</u>eligne - <u>G</u>elfand)

If $f: X \longrightarrow Y$ is a proper morphism

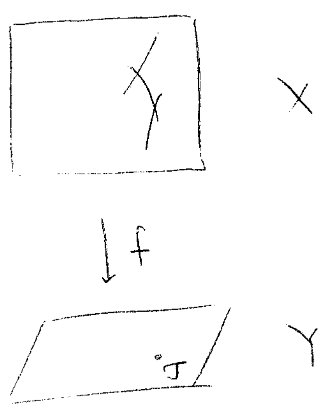
$$Rf_* IC_X \simeq \bigoplus IC_{Z_\ell}(L_\ell)[d_\ell]$$

where $\ell \in B$ for B finite, $Z_\ell \subset Y$ irreducible subvariety, L_ℓ a local system on Z_ℓ and $d_\ell \in \mathbb{Z}$.

Even if X is smooth this gives an interesting decomposition of $Rf_* \underline{\mathbb{Q}}_X$ as intersection cohomology sheaves. The original proof of this theorem uses ℓ-adic cohomology and so, does not yield any Hodge theoretic results. M. Saito in 1980's gave a complete generalization to the complex case, including the case of mixed Hodge structures.

"Moral example": let $f: X \to Y$, $g: T \to Y$ be maps from completely different varieties. The intersection cohomology of X still decomposes to the same local systems on Y

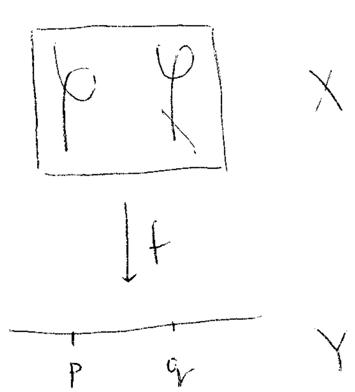
<u>Example</u>. Consider a morphism $f: X \to Y$ of surfaces w. an extra component over $\sigma \in Y$. Then



$$Rf_*\underline{\mathbb{Q}}_X \simeq IC_Y \oplus \mathbb{Q}_\sigma^{\#-1}[-2]$$

where $\#$ = number of irreducible components in the fiber

<u>Example</u> Let $f: X \to Y$ be a morphism from a surface to a curve with extra components over $p, q \in Y$.



Then

$$Rf_*\underline{\mathbb{Q}}_X \simeq \underline{\mathbb{Q}}_Y \oplus IC_Y(R^1) \oplus \mathbb{Q}_p[-2] \oplus \mathbb{Q}_q[-2]$$

Note that Decomposition theorem implies the invariant cycle theorem $H^1(X_0) \twoheadrightarrow H^1(X_\eta)$.

(2) Ngô's support theorem

A crucial part of the proof of this theorem is geometric and deals with the precise determination of the Z_ℓ's for the Hitchin map

$$h : M_D \longrightarrow \mathbb{A},$$

where M_D is the moduli space of stable Higgs bundles (E, φ), $\mathbb{A}$ is affine space, h is proper and flat of relative dimension d.

Let $\mathcal{U} \subset \mathbb{A}$ be the open set over which the fibers of h are reduced and consider

$$M_{\mathcal{U}} \xrightarrow{\ f\ } \mathcal{U},$$

then we have

<u>Theorem</u> (Ngô, '06) An irreducible subvariety $Z \subset \mathcal{U}$ appears in the decomposition if and only if $\exists Z^0 \subset Z$ such that $R^{2d} f_* \underline{\mathbb{Q}}\big|_{Z^0}$ is locally constant and Z is maximal.

Note that $R^{2d} f_* \underline{\mathbb{Q}}\big|_{Z^0}$ has finite monodromy because it is a linearization of a sheaf of finite sets.

(3) Perverse filtration and Lefschetz theorem.

Let $\pi : E \to B$ be a fiber bundle then we have the Leray filtration

$$H^*(E, \mathbb{Z}) \supset \operatorname{Im} H^*(B, \tau_{\leq \cdot} Rf_* \mathbb{Z}).$$

↗ truncation

In algebraic topology we may read this by taking a filtration by skeleta on B, i.e. an n-flag of closed subspaces

$$B_0 \subset B_1 \subset \dots \subset B_n = B$$

and pulling it back to $L_* = f^{-1} B_*$ on E.

The data on E_* gives you a filtration on cohomology $\ker HE \longrightarrow HE_*$ called the flag filtration. In general, the Leray and the flag filtrations are different, but we may choose a flag in such a way that the two filtrations become the same. It is enough to choose B_* so that

$$H^j(B_p, B_{p-1}, R^q \pi_* \mathbb{Z}) = 0$$

for all $j \neq p$ and all q. We say that such a B_* is π-allular.

This idea was adapted to algebraic varieties by Arapura in 2005. Given a projective morphism $f : X \to Y$ of quasi-projective varieties (for simplicity only assume $Y \overset{n}{\subset} \mathbb{A}^N$) then there is a flag of algebraic subspaces $Y_* \subset Y$ such that

$$\text{Leray filtration} = X_* \text{ filtration} \quad (\text{where } X_* = f^{-1} Y_*)$$

<u>Corollary</u> (M. Saito) Under these assumptions the Leray subspaces of the mixed Hodge structure $H(X)$ are mixed Hodge substructures

(this is the advantage of having algebraic skeleta)

Consider $\operatorname{Im} H(Y, \underbrace{{}^p\tau_{\le}}_{\text{perverse filtration}} Rf_* \mathbb{Z}) \subset H(X, \mathbb{Z})$

For any $C \in D_X$ (a certain full subcategory of the category of abelian sheaves on X); H, H_c (works for both) we have:

<u>Theorem</u> (de Cataldo - Migliorini)
For Y quasi-projective, $f : X \to Y$ any morphism there exists a filtration $Y_* \subset Y$ such that $X_* = f^{-1} Y_*$ is the perverse Leray filtration.

Lefschetz appears when you want to translate the condition of π-cellularity to this situation: we want our flag to satisfy

$$H^k(Y_p, Y_{p-1}, {}^pR^q f_* \mathbb{C}) = 0$$

$$\forall k \neq 0 \quad \forall q$$

(4) Character varieties

(Joint work in progress with Hausel and Migliorini) Let X be a curve of genus g,

$h : M_D \xrightarrow{h} \mathbb{A}$ smooth, quasi-projective, not affine.

Let M_B parametrize $\pi_1(X - \{pt\}) \longrightarrow GL_2(\mathbb{C})$

$$\gamma \longmapsto -\mathrm{Id}$$

Then M_B is affine, smooth and the generic fiber is an abelian variety. Hausel and Villegas make explicit the mixed Hodge structures of $H^*(M_B)$. Consider the multiplicative generators of the cohomology ring $\alpha \in H^2$, $\beta \in H^4$, $\psi_i \in H^3$ for $i = 1, .., 2g$. All of the generators have weight 4. They prove curious Hard Lefschetz: $\alpha^i : Gr_W HM_B \longrightarrow Gr_W HM_B$,

where W comes from α, β, ψ_i.

<u>Conjecture</u> $\Theta^*: HM_B \xrightarrow{\sim} HM_D$ exchanges the weight filtration with the perverse Leray filtration and the curious Hard Lefschetz with the relative Hard Lefschetz $\alpha^i : {}^pR^{-i}f_* \longrightarrow {}^pR^i f_*$.