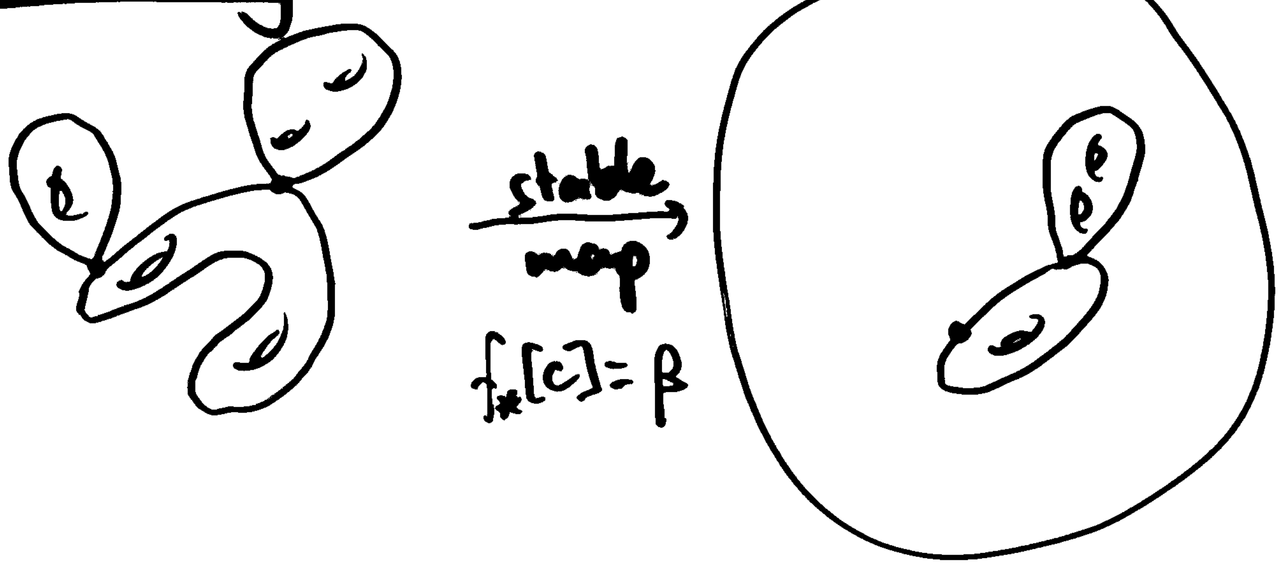
1.

BPS Invariants + Stable Pairs.

w/ Rahul Pandharipande

X CY 3fold throughout
$\Rightarrow$ vd $= 0$

1. GW theory (connected)



Invariants $N_{g,\beta} \in \mathbb{Q}$ $\qquad \beta \in H_2(X)$

2. GV theory (Count image curves?)

Conjecture $\exists\, n_{g,\beta} \in \mathbb{Z}$

- $g \geq 0$
- only finitely many $\neq 0$ for fixed β.

St. $N_{g,\beta}$ = function of $n_{r,\beta/d}$ and $n_{r,r}$

2.

[Gopakumar-Vafa,
Hosono-Saito-Takahashi
Schwarz-Shapiro
Toda]

try to define using torsion sheaves supported on image curves

$$\begin{array}{c} \overline{\mathrm{Jac}} \\ \downarrow \\ \mathcal{M}_{\substack{\text{curves} \subset X \\ \text{class } \beta}} \end{array}$$

projective fibration ~ Kähler product

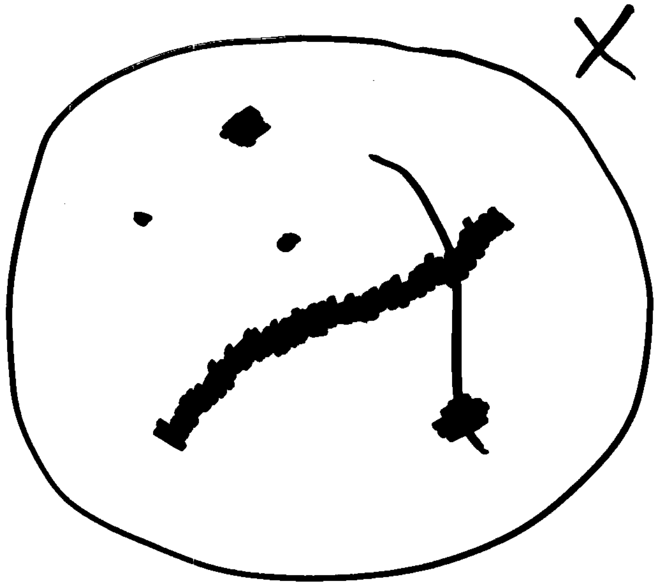
$$\rightsquigarrow \; sl_2 \times sl_2 \text{ Lefschetz action on } H^*(\overline{\mathrm{Jac}})$$

$$n_{g,\beta} \xleftarrow{\text{irreps}} \text{Lefschetz action on } H^*(\overline{\mathrm{Jac}})$$

"Count $H^*(T^{2g})$s in $H^*(\overline{\mathrm{Jac}})$"

[Joyce] – integrality directly in GW theory?

3. MNOP theory



Subschemes $\subset X$
in class β.

Deformation theory - not as subschemes
but as (ideal) sheaves
(fixed determinant)

$\Rightarrow$ Virtual cycle, dim 0

$\Rightarrow$ Invariants $\underline{I_{n,\beta} \in \mathbb{Z}}$ $n = \overset{1-g+\#\text{pts}}{\chi(\mathcal{O}_C)}$
$\beta = [C]$

MNOP conjecture $\{I_{n,\beta}\}_{\forall n} \longleftrightarrow \{N^\bullet_{g,\beta}\}_{\forall g}$ disconnected

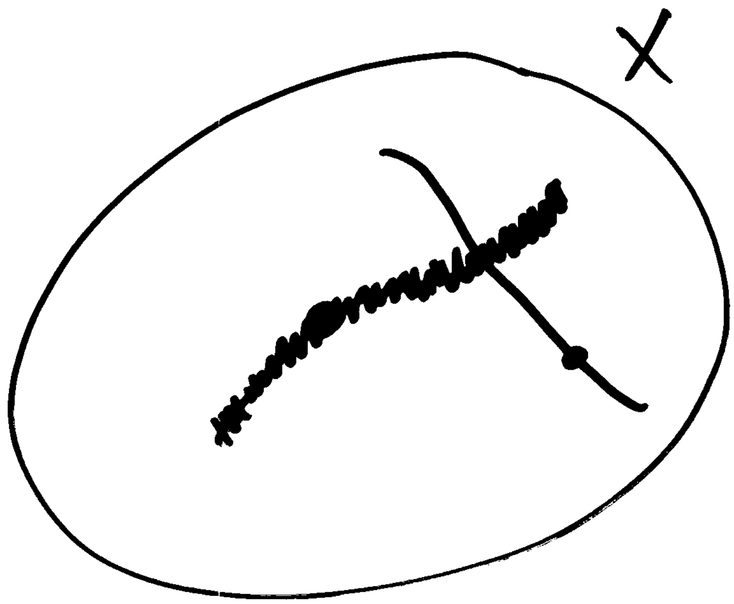
$$\boxed{\frac{\sum_n I_{n,\beta} q^n}{\sum_n I_{n,0} q^n} \overset{q=-e^{iu}}{=} \sum_g N^\bullet_{g,\beta} u^{2g-2}}$$

But free points, MNOP have to

4. New way to count embedded curves:

4.1 Stable pairs

~ curves + points on them
Cohen-Macaulay



Precisely: Sheaf F + section s $\Rightarrow$ Pair (F, s)

$[F] = \beta$

$\chi(F) = n$

$= 1 - g + \#(\text{points})$ if C reduced, irred.

Stability condition:
- F pure
- $\mathrm{coker}(s)$ dim 0.

$\Rightarrow$ Projective moduli space $P_n(X, \beta)$

if curves Gorenstein

5.

Deformation theory.

- Not $\mathrm{RHilb}^*(C)$
- Not pairs (F,s)
- Instead (fixed determinant) deformation theory of the complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X)$$

Thm Defs of $I^\bullet$ $\Rightarrow$ defs of $\mathcal{O}_X \xrightarrow{s} F$

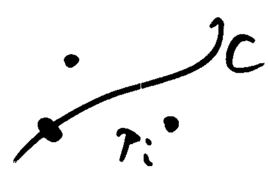
(eg $I^\bullet \underset{\text{q.iso.}}{\simeq} \mathcal{I}_C$ if (F,s) is $(\mathcal{O}_C, 1)$)

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \xrightarrow{s} F \to Q \to 0$$

zero dim'al, supported on pts on C
eg $\mathcal{O}_{p_i}$

$\Rightarrow$ Exact triangle $\underline{\mathcal{I}_C \to I^\bullet \to \mathcal{O}_{p_i}[1]}$ (*)

cf. MNOP ideal sheaves



free points
+ embedded points

$$0 \to \mathcal{I}_{C \cup \{p_i\}} \to \mathcal{I}_C \to \mathcal{O}_{p_i} \to 0$$

$\Rightarrow$ Exact triangle $\underline{\mathcal{O}_{p_i}[-1] \to \mathcal{I}_{C\cup\{p_i\}} \to \mathcal{I}_C}$ in $D^b(X)$

Deformation theory of $I^\bullet$;

Serre dual

$\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ deformations

$\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ obstructions

no higher obstructions

is <u>self-dual</u>.

Ie, morally, $P_n(X,\beta)$ is locally the critical locus of a hol'c f'n on a smooth ambient space.

- Holomorphic Chern-Simons f'nal.
- $\dfrac{\partial^2 CS}{\partial z_i \partial z_j} = \dfrac{\partial^2 CS}{\partial z_j \partial z_i}$
- $\nabla d = 0$

So if $P_n(X,\beta)$ smooth, Obs bundle $= T^*_{P_n(X,\beta)}$ and the invariant

$P_{n,\beta} = \#[P_n(X,\beta)]^{vir} \in \mathbb{Z}$ is $(-1)^{\dim} e(P_n(X,\beta))$

7. **Kai functions**

In general $P_{n,\beta}$ <u>not</u> just Euler char.

But (Kai Behrend) $\exists$ constructible fn
self-duality

$$\chi_P : P_n(X,\beta) \to \mathbb{Z}$$

$\chi_P = (-1)^{\dim P_n(X,\beta)}$ at smooth points

such that $\underline{P_{n,\beta} = e\left(P_n(X,\beta), \chi_P\right)}$

[MNOP = Stable-pairs conjecture: | Joycian wall crossing [Joyce] [Kontsevich-Soibelman] [Denef-Moore-Diaconescu]

$$I_{n,\beta} = \sum_m P_{m,\beta}\, I_{n-m,0}$$
]

Hilb

locus where m pts lie on C

n-m pts on X\C

8.

GV = GW = MNOP = Stable pairs

<u>all</u> conjectural!

gives the following GV = Stable pairs conjecture.

β irreducible case: If $\beta = \beta_1 + \beta_2$ then one of β_1, β_2 is zero.

curve classes: β_1, β_2

$\Rightarrow$ CM curves C with $[C] = \beta$ are reduced and irreducible.

$$Z_{P,\beta}(q) := \sum_n P_{n,\beta}\, q^n$$ is the Laurent series of a rational f'n of q, inv't under $q \leftrightarrow \frac{1}{q}$,

and $$\underline{Z_{P,\beta}(q) = \sum_{g \geq 0} n_{g,\beta}\, q^{1-g}(1+q)^{2g-2}}$$ a <u>finite</u> sum.

$$(q^{-1}+2+q)^{g-1}$$

9.

More generally any Laurent series

$$Z(q)=\sum a_n q^n$$

has a finite BPS form

$$\sum_{g=0}^{G} n_g\, q^{1-g}(1+q)^{2g-2}$$

iff $\underline{a_n = a_{-n} + (-1)^{n-1} nC}$ $\forall n$

$\underline{a_n = 0}$ $\quad n \le -G.$

10.

$$P_n(X,\beta) \xrightarrow{\text{supp}} \mathrm{Chow}(X,\beta)$$

parametrises curves in class β.

Can compute $e\big(P_n(X,\beta), \chi_P\big)$ by "pushing down to Chow" take Euler chars of fibres $P_n(C)$, weighted by χ_P

{stable pairs supp. on C}

to get constructible fn $P_{n,C} \leftarrow\!\shortmid C$ on $\mathrm{Chow}(\beta)$.

s.t. $P_{n,\beta} = e\big(\mathrm{Chow}(X,\beta), P_{n,C}\big)$

So can work locally, and talk about contribution of a single curve C.

Smooth curve case.

$C \subseteq X$ smooth, genus g.

- $P_{n<1-g}(C) = \emptyset \quad \Rightarrow \underline{P_{n<1-g,C} = 0}$
- $P_{1-g}(C) = 1\,\text{pt} = \{(\mathcal{O}_C, 1)\} \quad \Rightarrow \underline{P_{1-g,C}} = 1 \cdot \chi_p(\mathcal{O}_C, 1)$
- $P_{2-g}(C) = C \quad \Rightarrow P_{2-g,C} = -e(C) = \underline{(2g-2)}$ (up to χ_p fns!)
- $P_{3-g}(C) = S^2(C)$

$\vdots$

- $P_{1-g+n}(C) = S^n(C) \quad P_{1-g+n,C} = \underline{(-1)^n e(S^n C)}$ modulo χ_p-fns

But: $\underline{\sum (-1)^n e(S^n C)\, q^{1-g+n} = q^{1-g}(1+q)^{2g-2}}$

contribution of $n_{g,C}$

i.e.

12. χ_P stuff?

Miracle: $P_n(X,\beta) \xrightarrow{\phi_n} \mathcal{M}_n(X,\beta)$

(stable) sheaves F

$[F]=\beta,\ \chi(F)=n$

Thm: $\chi_P = (-1)^{n-1}\phi_n^*\chi_{\mathcal{M}}$.

Clear for large $n \gg 0$: ϕ_n is $\mathbb{P}(H^0(F)) \cong \mathbb{P}^{n-1}$ bundle

(True thm, even when ϕ_n not onto).

And invt under tensoring with line bdls.

So for smooth curves find

$$Z_{P,C}(q) = \chi_{\mathcal{M}}(\mathcal{O}_C)\, q^{1-g}(1+q)^{2g-2}$$

ie $\begin{cases} n_{g,C} = \chi_{\mathcal{M}}(\mathcal{O}_C) \\ n_{r,C} = 0 \quad r \neq g. \end{cases}$

13.

Singular curves

Read off $n_{r,C}$s inductively from BPS/stable pairs conjecture.

- $P_{<1-g}(C) = \emptyset$
- $P_{1-g}(C) = pt = \{(\mathcal{O}_C, 1)\}$ $\Rightarrow n_{g,C} = 1.\left|(-1)^g \chi_\mu(\mathcal{O}_C)\right|$
- $P_{2-g}(C) \cong C$
  eg if C Gorenstein

and $P_{2-g,C} = (2g-2)n_{g,C} + n_{g-1,C}$

ie the $n_{g,C}$ genus g curves already counted are expected to contribute $(2g-2) = e(\Sigma_g)$ to $P_{2-g,C}$. But $P_{2-g}(C) \cong C$ may not have $e = 2g-2$.

(Worse – χ fns)

So define $n_{g-1,C}$ to be this discrepancy

$$P_{2-g,C} + (2g-2)n_{g,C}$$

Serre duality

$$F \longmapsto F^\vee := \mathcal{E}xt^2_X(F, K_X)$$

$$\mathcal{M}_n(X,\beta) \longleftrightarrow \mathcal{M}_{-n}(X,\beta)$$

$F = L_C$ line bundle on smooth curve

$\Rightarrow F^\vee = L_C^* \otimes K_C$.

Preserves χ_μ !

Fibre of $\phi_n : P_n(X,\beta) \to \mathcal{M}_n(X,\beta)$ over F

is $\underline{\mathbb{P}(H^0(F))}$, $\underline{e = h^0(F)}$

Fibre of ϕ_{-n} over $F^\vee$ is $\mathbb{P}(H^0(F^\vee)) = \underline{\mathbb{P}(H^1(F)^*)}$, $\underline{e = h^1(F)}$

$\Rightarrow$ Difference of their contributions is

$$(-1)^{n-1}\chi_\mu(F)\big(h^0(F) - h^1(F)\big) = \underline{(-1)^{n-1} n\, \chi_\mu(F)}$$

Take $e(\cup \mathcal{M}_n, -) \Rightarrow P_{n,\beta} - P_{-n,\beta} = (-1)^{n-1} n\, e(\mathcal{M}_n(X,\beta), \chi_\mu)$

9.

More generally any Laurent series

$$Z(q) = \sum a_n q^n$$

has a finite BPS form

$$\sum_{g=0}^{G} n_g \, q^{1-g} (1+q)^{2g-2}$$

iff $\underline{a_n = a_{-n} + (-1)^{n-1} nC}$ $\forall n$

$\underline{a_n = 0}$ $\quad n \leq -G.$

RICHARD THOMAS: COUNTING CURVES IN 3-FOLDS

NOTES BY KARTIK VENKATRAM

Let $C \subset X^n$ be a smooth embedded curve of class $\beta \in H_2(X)$: its deformations are given by $\mathrm{Def}(C) = H^0(\nu_C)$ and the obstructions to deformation by $\mathrm{Obs}(C) = H^1(\nu_C)$, giving a *virtual dimension* of the associated moduli space of $h^0 - h^1 = c_1(X)\beta + (n-3)(1-g)$. When counting curves, you would like the answer to be deformation invariant: to get a finite number, then, one must impose a number of conditions equal to the virtual dimension on a compactified moduli space, and obtain the count by intersecting *virtual cycles*. For the context of this talk, we will simplify to using Calabi-Yau 3-folds, whose moduli space has virtual dimension 0.

(1) Gromov-Witten theory considers parameterized *stable maps* $f : C \to X$, where the curve is almost smooth (only nodal singularities) but f can be far from an embedding (certain components contracted, other components many-to-one). The stability condition, though, implies that it has no infinitesimal automorphisms, thus giving a reasonable compactified moduli space (projectified Deligne-Mumford stack). One obtains a set of invariants $N_{g,\beta} \in \mathbb{Q}$.

(2) These rational numbers, though, should not be an integral part of the theory. For instance, given a curve mapping in two-to-one, the contribution to the Gromov-Witten invariant is $\frac{1}{2}$, but the embedded curve should have contribution 1. Thus, Gopakumar-Vafa conjecture that one can count "'image curves"' and obtain numbers $n_{g,\beta}$, only finitely many of which are nonzero for fixed β, s.t. $N_{g,\beta}$ is a function of $n_{r,\beta/d}$ and n, r, $(r \leq g, d \geq 1), (r \leq g, \gamma \leq \beta)$.

    [G-V,Hsono-Saito-Takahashi, Schwartz-Shapiro, Toda] try to define this using torsion sheaves supported on image curves. One has $\tilde{Jac} \to \mu_{\text{curves } \subset X \text{ of class } \beta}$ and each image curve should 1. Relating projective fibrations to Kähler products and $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, the Lefschetz action on $H^*(\tilde{\mathrm{Jac}})$ gives these number $n_{g,\beta}$. Naively, one counts $H^*(T^{2g})$s in $H^*(\tilde{\mathrm{Jac}})$. [Joyce] attempts to show integrality directly in GW theory, while [Us] is closer to the approach of [Katz-Kleman-Vafa].

Example. If one has a genus g embedded curve with branches, taking a family of such curve might limit in a curve with branches coming together. In GW theory, the limit curve would be the normalization of that curve, while in GV theory one would simply take the given image curve as being a curve of higher genus. If one wants to restrict to a particular genus, one thus must work on an open moduli space (i.e. not all limits exist).

(3) MNOP theory: we consider subschemes Z of X in class β. We consider the deformation theory of these not as subschemes but as ideal sheaves with fixed determinant. We obtain virtual cycles which are always of dimension 0, and then invariants $I_{n,\beta} \in \mathbb{Z}$ ($n = \chi(\mathcal{O}_Z), \beta = [C]$). The MNOP conjecture states that we have a correspondence between $\{I_{n,\beta}\}_{\forall n}$ and $\{N^*_{g,\beta}\}_{\forall g}$ via an equivalence between generating functions

$$\frac{I_{n,\beta}q^n}{I_{n,0}q^n} = \sum_g N^*_{g,\beta} u^{2g-2} \tag{1}$$

But MNOP has to correct for free points: taking our example above, the limit of the singularity in the Hilbert scheme (in local coordinates) is $(y,z)(x,z-t) = (xy, xz, y(z-t), z(z-t)) \to (xy, xz, yz, z^2)$ as $t \to 0$, which is different from the node (xy, z) as it has an extra free point.

(4) There is a new way to count embedded curves using stable pairs, which are roughly curves (Cohen-Maucalay) with points on the curve. More precisely, we have a sheaf F and a section s such that $[F] = \beta, \chi(F) = n = 2-g+\#(\text{points})$ if C is reduced, irreducible. The stability condition says that F is pure and Coker (s) has dimension 0. The sheaf is made as an extension of the structure sheaf of the curve by the structure sheaf of the extra points. One obtains a projective module space $P_n(X,\beta)$ (if the curves are Gorenstein, the moduli space is a relative Hilbert scheme $RHilb^{g-1+n}(C)$): examples of points are $(F = \mathcal{O}_C, 1), (\mathcal{O}_C(p_i), s_{p_i})$. Returning to the previous example, taking C_1, C_2 to be the branches with limit point p, the pairs limit will be $(F,s) = (\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}, (1,1))$, where

$$0 \to \mathcal{O}_C \xrightarrow{(1,1)} \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_p \to 0 \tag{2}$$

We perform the (fixed determinant) deformation theory on the complex

$$I^* = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X) \tag{3}$$

Theorem 1. *Deformations of I^* give deformations of $\emptyset_X \to F$.*

For instance, I^* is quasi-isomorphic to $\mathcal{I}_C$ if $(F,s) = (\mathcal{O}_C, 1)$. In general, we obtain $0 \to \mathcal{I}_C \to \mathcal{O}_X \to F \to Q \to 0$, where Q is zero-dimensional and supported on points on C (e.g. $\mathcal{O}_{p_i}$). We obtain an exact triangle in the derived category $II_C \to I^* \to \mathcal{O}_{p_i}[-1]$ of MNOP ideal sheaves. On the other hand, taking $0 \to \mathcal{I}_{C\cup\{p_i\}} \to \mathcal{I}_C \to \mathcal{O}_{p_i} \to 0$ gives $\mathcal{O}_{p_i}[-1] \to \mathcal{I}_{C\cup p_i} \to \mathcal{I}_C$ which reverses the extension of the preivious one. One should see this from the perspective of vector bundle stability: a vector bundle is stable iff () based on slopes. The triangles above are simply exact

Naively, you want to move in the space of stability conditions (like Bridgeland's space of stability conditions) and cross a wall such that on one side the first triangle is unstable and on the other side the second one is. On one side is the moduli space of derived objects, and on the other side stable pairs. We have a Serre duality $\mathrm{Ext}^1(I^*, I^*)_0$ (deformations) and $\mathrm{Ext}^2(I^*, I^*)_0$ (obstructions, and no higher), giving a self-dual obstruction theory. Morally, $P_n(X, \beta)$ is lcoally the critical locus of a holomorphic function on a smooth ambient space.

So if $P_n(X, \beta)$ is smooth, bundle $= T^*_{P_n(X,\beta)}$ and the invariant $P_{n,\beta} := \#[P_n(X, \beta)]^{vir} \in \mathbb{Z}$ is $(-1)^{\dim}\ e(P_n(X, \beta))$.

(5) Kai functions. In general, $P_{n,\beta}$ are not just the Euler classs, but via Behrend self-duality $\exists$ a constructible function $\chi_p : \phi_n(X, \beta) \to \mathbb{Z}$ (which iss simply $(-1)^{\dim}$ at smooth points) such that $P_{n,\beta} = e(P_n(X, \beta, \chi_\beta))$. We obtain an equivalence between MNOP, stable pairs conjective, and Joycian wall crossing, and we obtain our original invariants $I_{n,\beta} = \sum_m P_{m,\beta} I_{n-m,0}$. This gives the following GV = stable pairs conjecture: for β an irreducible curve, $Z_{p,\beta}(q) = \sum_n P_{n,\beta} q^n$ is the Laurent series of a rational function of q invariant under $q \to \frac{1}{q}$. and $Z_{p,\beta}(q) = \sum_{g \geq 0} n_{g,\beta} q^{1-g}(1+q)^{2g-2}$ a finite sum.

...

(6) In the smooth curve case $C \subset X$ of genus g, we have
- $P_{n<1-g}(C) = \varnothing \implies P_{n \leq 1-g,C} = 0$
- $P_{1-g}(C) = 1\mathrm{pt} = \{(\mathcal{O}_C, 1)\} \implies P = 1$
- $P_{2-g}(C) = C \implies P = -e(C) = 2g - 2$
- $P_{3-g}(C) = S^2(C)$
- $P_{1-g+n}(C) = S^n(C)$, and $P_{1+g+n,C} = (-1)^n e(S^n C)$ modulo χ_p-functions.

But $\sum (-1)^n e(S^n C) q^{1-g+n} = q^{1-g}(1+q)^{2g-2}$.

(7) χ_p stuff? The miracle is that we have $\phi_n : P_n(X, \beta) \to MM_n(X, \beta)$ and $\chi_p = (-1)^{n-1} \phi_n^* \chi_n$. This clear for large $n >> 0$ and true in general (and invariant under tensoring with line bundle). For smooth curves, we find $Z_{P,C}(q) = \chi_{\mathcal{M}}(\mathcal{O}_C) q^{1-g}(1+q)^{2g-2}$, i.e. $n_{g,c} = \chi_{\mathcal{M}}(\mathcal{O}_C), n_{r,C} = 0$ for $r \neq g$.

Read off $n_{r,C}$ inductively from ... conjecture. By the above series, in the 2 case we find that in the $n_{g,C}$ the genus g curves are already counted and are expected to contribute $2g - 2 = e(\Sigma_g)$ to $P_{2-g,C}$. But $P_{2-g} \cong C$ be not have $e = 2g - 2$. So we define $n_{g-1,C}$ to be this discrepancy $P_{2-g,C} + (2g-2) n_{g,C}$. Serre duality gives $F \to F^\vee := \mathcal{E}xt^2_X(F, K_X)$, $\mathcal{M}_n(X, \beta) \to \mathcal{M}_{-n}(X, \beta)$. This preserves $\chi_{\mathcal{M}}$. The fiber of $\phi_n : P_n \to \mathcal{M}_n$ over $\mathbb{F}$ is $\mathbb{P}(H^0(F)), e = h^0(F)$ and that of ϕ_{-n} is $\mathbb{P}(H^0(F^\vee)) = \mathbb{P}(H^1(F)^*), e = h^1(F)$. The difference of their contributions is $(-1)^{n-1} \chi_{\mathcal{M}}(F)(h^0(F) - h^1(F)) = (-1)^{n-1} n \chi_{\mathcal{M}}(F)$. Take

$e(\bigcup \mathcal{M}_n, -) \implies P_{n,\beta} - P_{-n,\beta} = (-1)^{n-1} n e(\mathcal{M}_n(X,\beta), \chi_{\mathcal{M}})$. Similarly, $e(\bigcup \mathcal{M}_n(C), -) \implies P_{n,C} - P_{-n,C} = (-1)^{n-1} n e(\mathcal{M}_n(C), \chi_{\mathcal{M}}|_{\mathcal{M}(C)})$. More generally, any Laurent series $Z(q) = \sum a_n q^n$ has a finite BPS form $\sum_{g=0}^{G} n_g q^{fg} (1+q)^{2g-2}$ iff $a_n = a_{-n} + (-1)^{n-1} nC$ for all n, $a_n = 0$ for $n \leq -G$.