DANIEL HUYBRECHTS: DERIVED CATEGORIES AND CHOW GROUPS OF K3 SURFACES

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Throughout the talk, X will be a smooth, projective K3 surface over a field K which is either $\overline{Q}$ or $\mathbb{C}$: its distinguishing features are that $\omega_X \cong \mathcal{O}_X$ generated by σ_X and $H^1(X, \mathcal{O}_X) = 0$. The whole complexity of the theory can be realized in one example of a K3, namely $X \subset \mathbb{P}^3, \sum_{i=0}^3 x_i^4 = 0$. The Chow ring $CH(X)$ splits as $CH^0 \oplus CH^1 \oplus CH^2$, where $CH^0 = \mathbb{Z}, CH^1 = \mathrm{Pic}(X)$. CH^2 is the interesting bit, and is the space of points in the surface moduli linear equivalence. We have a surjective degree map $CH^2 \to \mathbb{Z}$, and call the kernel $A(X)$.

Theorem 1 (Mumford). *If $K = \mathbb{C}$, $A(X)$ is huge (beyond infinitely generated).*

Conjecture 1 (Bloch-Beilinson). *For $K = \overline{\mathbb{Q}}$ (or any number field), $A(X)$ is trivial (or torsion over a number field).*

In the case $K = \mathbb{C}$, the singular cohomology gives $H^*(X, \mathbb{Z}) = H^0 \oplus H^2 \oplus H^4 = \mathbb{Z} \oplus H^2 \oplus \mathbb{Z}$ equipped with an intersection pairing and Hodge structures on each part (weight $2i$ on H^{2i}). Note that the Hodge structure on H^2 gives $H^{1,1}(X) \cap H^2(X, \mathbb{Z}) = \mathrm{Pic}(X)$. In order to use this to analyze $A(X)$, we modify these slightly: first modify the intersection pairing to the Mukai pairing (H^2 with the usual intersection pairing, $H^0 \oplus H^4$ with negative the usual pairing). We further modify the Hodge structure with a weight 2 one on

$$(1) \quad H^* = \tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^4 \implies \tilde{H}^{1,1}(X) = \Im(CH^* \to H^*) \cap H^*(X, \mathbb{Z})$$

We obtain a *transcendental lattice* $T(X) = \tilde{H}^{1,1}(X, \mathbb{Z})^\perp \subset H^2(X, \mathbb{Z})$ s.t. $f : X \xrightarrow{\sim} X$ gives $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ called the *Hodge isometry*: we obtain an induced automorphism of the lattice.

Conjecture 2 (Bloch). *If f^* is the identity on the lattice (i.e. f^* respects the regular two-form σ), then f^* is the identity on CH^2.*

If we know it acts trivially on all of cohomology, a version of global Torelli gives the conclusion.

This related to the algebraic geometry of the self-product of X. Look at the space E of Hodge endomorphisms of $T(X)_{\mathbb{Q}}$. By a theorem of Zahim, E is a field, and it totally real or complex multiplication. Embedding E in $H^{2,2}(X \times X, \mathbb{Q})$, we find that classes in E are algebraic $\Leftrightarrow$ the Hodge conjecture is true for $X \times X$.

Returning to the study of the Chow ring of X, we now look at category of coherent sheaves on X. It is a K-linear abelian category, and in fact determines X. To make it more flexible, we look at the bounded derived category of coherent sheaves $D^b(X)$, which is K-linear triangular and does not determine X. I have a map $D^b(X) \to CH(X)$ given by mapping a complex E^* to its Chern character $\mathrm{ch}(E^*)$. Look at the group $\mathrm{Aut}D^b(X)$ of equivalences $\phi : D^b(X) \xrightarrow{\sim} D^b(X)$. We have a map ρ from this space to $\mathrm{Aut}\tilde{H}(X, \mathbb{Z})$, the group of Hodge isometries (specifically, Mukai showed that from ϕ one obtains an isometry ϕ^H. Furthermore, by [Orlov] $\Im(\rho)$ is a subgroup of index ≤ 2, i.e. almost every Hodge isometry can be lifted.

Join work with Macri and Stellani has shown that $\Im(\rho)$ in fact has index 2, and $\Im\rho = \mathrm{Aut}_{\oplus}\tilde{H}(X, \mathbb{Z})$. The techniques that go into it involve the deformation theory of complexes and nonprojective K3 surfaces). Once we have described the image of this representation, the next question is about the kernel.

Conjecture 3 (Bridgeland)**.** *Ker ρ is big (it has many* spherical objects*).*

Here, $E^* \in D^b(X)$ is spherical if $\mathrm{Exl}^*(E, E) = H^*(S^2, K)$. Then T_E gives a spherical twist in $\mathrm{Aut}D^b(X)$ and $T_E^2 \in \mathrm{Ker}\ \rho$.

We want to use a similar strategy on the Chow group. Define $\rho^{CH} : \mathrm{Aut}D^b(X) \to \mathrm{Aut}(CH(X)), \phi \mapsto \phi^{CH}$ (which does not respect degree or ring structure, but does respect the additive structure). One can show that

Theorem 2. Ker $\rho^{CH} =$ Ker ρ^H.

This confirms a part of Bloch's conjecture above, and is heavily based on [HMS] which uses nonprojective K3 surfaces. By Beauville-Voisin, for $X/\mathbb{C}$ the subring $R(X) = \langle ch(L) = 1 + c_1 + \frac{c_1^2}{2} | L \in \mathrm{Pic}X \rangle \subset CH(X)$ is precisely $\tilde{H}^{1,1}(X, \mathbb{Z})$. Moreover, for all $c_X \in CH^2(X)$, $\forall L \in \mathrm{Pic}(X), c_1^2(L) \in \mathbb{Z}c_X$ and if $x \in C \subset X$, $[x] = C_X$ is a rational curve. Applying this to the BB conjecture, we see that for $CH(X) \cong R(X) \subset CH(X_{\mathbb{C}}) \twoheadrightarrow \tilde{H}^{1,1}(X, \mathbb{Z})$, $R(X) \cong \tilde{H}^{1,1}(X, \mathbb{Z})$, and we need to show $\alpha \in CH^2(X) \implies \alpha \in R(X)$, i.e. $\alpha \in \mathbb{Z}c_X$.

How do we produce cycles? The geometric way is to take $x \in X/\overline{\mathbb{Q}}$, and ask if $\forall x$, $\exists C \subset X$ rational with $x \in C, [x] = C_X$. Using Chern characters, one can instead ask if $c_2(E) \in \mathbb{Z}c_X$ for $E \in D^b(X)$.

To test the BB conjecture, we try to:

- Produce many classes in $CH^2(X/\overline{Q})$.
- Show that they are all contained in $\mathbb{Z}c_X$.

The easy part: for $X/\overline{Q}$, $X/\mathbb{C}$ the induced K3, $E \in D^b(X_{\mathbb{C}})$, we have $E^* \in D^b(X/\overline{Q})$ (giving $c_2(E) \in CH^2(X/\overline{Q})$).

Theorem 3. *For $E^* \in D^0(X)$ spherical, $c_2(E) \in Zc_X$ (so* $\mathrm{ch}(E) = R(X)$*).*

The proof of this uses deformation theory to reduce to vector bundles, and then use a classical result of Lazarsfeld saying that curves in K3 surfaces are Brill-Noether general.

If I have $X/\overline{Q}$, then any equivalence $\phi : D^b(X_{\mathbb{C}}) \xrightarrow{\sim} D^b(X_{\mathbb{C}})$ is defined over $\overline{\mathbb{Q}}$.

Theorem 4. *For X defined over $\overline{Q}, \mathbb{C}$, $\phi : D^b(X) \xrightarrow{\sim} D^b(X)$ gives $\phi^{CH} : CH(X) \xrightarrow{\sim} CH(X)$, and ϕ preserves $R(X)$.*