MARTIN OLSSON: MAIN COMPONENTS OF MODULI SPACES AND LOG GEOMETRY

NOTES BY KARTIK VENKATRAM

Motivation: we want to compactify moduli space $\mathcal{M} \to \overline{\mathcal{M}}$. The case of moduli spaces of curves is misleading, because they are too nice. We have two coices: study $\overline{\mathcal{M}}$ or $\mathcal{M}$ if you can take a good "closure" of $\mathcal{M}$ without including bad components of $\overline{\mathcal{M}}$.

Example. The Tate curve:

(1)
$$\begin{tikzcd}
\text{smooth elliptic curve} \arrow[r] \arrow[d] & E_q \arrow[d] & \text{nodal curve} \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[[q]][1/q] \arrow[r] & \operatorname{Spec} \mathbb{Z}[[q]] & \operatorname{Spec} \mathbb{Z} \arrow[l, "q=0"']
\end{tikzcd}$$

This is a family of elliptic curves whose special fiber is a nodal curve (right side) and whose general fiber is smooth elliptic (left side).

1. Monoids

Let $P = \operatorname{Cone}(1, \mathbb{R})_{\mathbb{Z}} \subset \mathbb{R} \times \mathbb{R}, P \ltimes \mathbb{N}$ the free monoid on generators x_n and q modulo $x_{n+2} + x_n = q2x_{n+1}$. This is a graded monoid, where we set $\deg\, x_1 = 1, \deg\, q = 0$.

Now

(2)
$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \operatorname{Proj}\ (\mathbb{Z}[P \ltimes \mathbb{N}]) \arrow[d] & \text{chain of curves} \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[q^{\pm}] \arrow[r] & \operatorname{Spec} \mathbb{Z}[q] & \operatorname{Spec} \mathbb{Z} \arrow[l]
\end{tikzcd}$$

where $\mathbb{Z}$ acts on the middle term of the top row by $x_i \to x_{i+1}$. Taking a quotient by the $\mathbb{Z}$-action gives

(3)
$$\begin{tikzcd}
\text{elliptic curve} \arrow[r] \arrow[d] & E_q \arrow[d] & \text{nodal curve} \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[[q]][1/q] \arrow[r] & \operatorname{Spec} \mathbb{Z}[[q]] & \operatorname{Spec} \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Question: how do you see the toric structure on E_q? The answer is to use a log structure.

2. Log Structure (Fontaine, Illusie, Kato, ...)

For a scheme, let $\mathrm{Div}(X)$ be the groupoid of pairs (L, s), where L is a line bundle on X and $s : X \to \mathcal{O}_X$. Maps $f : Y \to X$ induce maps $f^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$, giving a Div stack on $X_{\text{ét}}$.

Example. For P a monoid, consider $X = \operatorname{Spec} k[P]$ and $F : P \to \mathrm{Div}, p \mapsto (\mathcal{O}_X \xrightarrow{p} \mathcal{O}_X)$. This gives a map $\overline{\mathcal{M}} = P/F^{-1}(\mathrm{id}_{\mathcal{O}_X}) \to \mathrm{Div}$.

Definition 1. *A* log structure *on X is a pair $\mathcal{M}, \alpha$ where $\mathcal{M}$ is a sheaf of monoids and α is a map $\mathcal{M} \to \mathcal{O}_X$ s.t. $\alpha(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}^*$.*

Now, given $(\mathcal{M}, \alpha)$, set $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}_X^* \xleftarrow{\pi} \mathcal{M}$: for $m \in \mathcal{M}, \pi^{-1}(m)$ has a $\mathcal{O}_X^*$ aaction, giving a line bundle $L \xrightarrow{\alpha} \mathcal{O}_X$. This gives a category of log structures, where the morphisms are of sheaves of monoids compatible with the maps to the structure sheaf. Any map $Y \to X$ has an associated f^* from log structures on X to those on Y.

Definition 2. *A* log scheme *is a pair (X, M_X) where X is a scheme, M_X a log structure. A map of log schemes $f : (Y, M_Y) \to (X, M_X)$ is a map $Y \to X$ along with a pullback $f^b : f^* M_X \to M_Y$.*

In the commutative diagram above, replacing E_q with (E_q, M_{E_q}) and $\operatorname{Spec} \mathbb{Z}[[q]]$ with $(\operatorname{Spec} \mathbb{Z}[[q]], M_q)$ gives an example of log smooth morphism.

2.1. Aside on $\mathcal{A}_{g,d}$.

For S a scheme, we set $\mathcal{A}_{g,d}$ to be the set of pairs (A, λ) where A/S is an abelian scheme of dimension g, $\lambda : A \to A^t$ is a polarization, and $\mathrm{rkKer}\ (\lambda) = d^2$.

Remark. For $d = 1$, Alexeev constructs $\mathcal{A}_g \to \overline{\mathcal{A}}_g^{Alex}$.

Now, let $\tau_{g,d}$ classify triples (A, P, L), where

- A/S is an abelian scheme of dimension g.
- P is an A-torsor with associated map $f : P \to S$.
- L is an ample line bundle on P s.t. $f_* L$ has rank d.

We obtain a group $_{(P,L)} = \{\sigma : P \xrightarrow{\sim} P \text{ an A-equivalence}, \tau : \sigma^* L \xrightarrow{\sim} L\}$ which fits into the diagram

$$1 \to \mathbb{G}_m \to {}_{(P,L)} \to \mathrm{Ker}\ (\lambda_L : A \to P^t = A^t) \to 1 \tag{4}$$

In stack terms, the map $\tau_{g,d} \to \mathcal{A}_{g,d}$ is *rigidification* by .

Remark. If $d = 1$, we have a section $s : A_{g,d} \to \tau_{g,d}$.

2.2. **Degenerations.** For V a DVR, $K = \mathrm{Frac}(V)$, $(A_K, P_K, L_K) \in \tau_{g,d}(K)$, we obtain an induced triple $(G, G \circlearrowright (P, M_p) \to (\mathrm{Spec}\ V, M_V), L)$ where G is a semiabelian scheme over V, $(P, M_p) \to (\mathrm{Spec}\ V, M_V)$ is log smooth and proper, and L is a line bundle on P. Let $\Delta = (M_V, (P, M_P) \to (V, M_V), L, G)$, ${}_{\Delta} = \mathrm{Aut}_G((P, M_P, L)/(V, M_V))$. Then we have a commutatiive diagram

$$1 \to \mathbb{G}_m \to {}_{\Delta} \to N \to 1 \tag{5}$$

where N is a flat group scheme of rank d^2. Let $\overline{\tau}_{g,d}$ classify the log version, i.e. $(M_S, G, (P, M_P) \to (S, M_S), L)$. We obtain a diagram

$$\begin{tikzcd} \tau_{g,d} \arrow[r] \arrow[d] & \overline{\tau}_{g,d} \arrow[d] \\ \mathcal{A}_{g,d} \arrow[r] & \overline{\mathcal{A}}_{g,d} \end{tikzcd} \tag{6}$$

where the right hand map is rigidification by the theta group.

Theorem 1. *$\overline{\mathcal{A}}_{g,d}$ is proper over $\mathbb{Z}$, log smooth over $\mathbb{Z}[1/d]$, and $\mathcal{A}_{g,d} \to \overline{\mathcal{A}}_{g,d}$ is dense open.*

Remark. If the log structure is trivial, log smoothness is equivalent to having toric singularities. For $d = 1$, $\overline{\mathcal{A}_g}$ is the normalization of MC in $\overline{\mathcal{A}}_g^{Alex}$.

3. Theta level structure (Mumford)

Let $\delta = (d_1, \ldots, d_g), d = d_1 \cdots d_g$ over $\mathbb{Z}[1/2d]$. Now, set $K(\delta) = \bigoplus \mathbb{Z}/(d_i)$ with dual $K\hat{(\delta)} = \mathrm{Hom}(K(\delta), \mathbb{G}_m)$, and define $(\delta) = \mathbb{G}_m \times K(\delta) \times K\hat{(\delta)}$, with product $(\alpha, x, \ell)(\alpha', x', \ell') = (\alpha\alpha'\ell(x'), x+x', \ell+\ell')$. Let $\mathcal{M}_{g,\delta}(S)$ classify quadruples $(A, L, \sigma, \epsilon : e^*L \xrightarrow{\sim} \mathcal{O}_S)$ s.t.

- A/S is an abelian scheme,
- L is relatively ample on A and symmetric,
- $\sigma : (\delta)_S \xrightarrow{\sim} {}_{A,P,L}$ is the identity on $\mathbb{G}_m$.

By a theorem of Mumford, if $\delta | d_i$ for all i, then $\mathcal{M}^{tot}_{g,d}$ is a q-projective variety. The reason is that there is a unique irreducible representation V_δ of (δ) of rank d on which $\mathbb{G}_m$ acts by multiplication:

$$\begin{tikzcd} A \arrow[r] & \mathbb{P}(f_*L) \arrow[r, "\sim"] & \mathbb{P}(V_{\delta,S}) \\ S \arrow[u, "e"'] & & \end{tikzcd} \tag{7}$$

giving $\mathcal{M}^{tot}_{g,\delta} \to \mathbb{P}V_\delta$.

Question: How do we resolve the singularities of the closure?

Answer: There is a log smooth resolution $\overline{\Theta}^{tot}_{g,\delta}$ of $\overline{M}^{tot}_{g,\delta}$.

Finally, setting $\Sigma_{g,\delta} = \mathrm{Isom}(, (\delta))$, we obtain

$$\begin{tikzcd}
P \arrow[r, "\sim"] \arrow[dr] & P^2 = P \arrow[d] \\
 & \Sigma_{g,\delta} \arrow[d] \\
 & \tau_{g,d}
\end{tikzcd} \tag{8}$$

where $\mathcal{M}_{g,\delta}$ is a rigidification of the fixed locus of the top map.

Martin Olsson 1/29/09

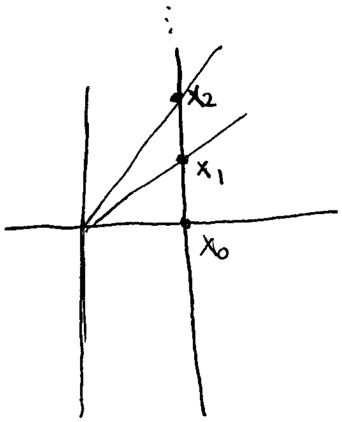
Motivation: $\mathcal{M} \to \overline{\mathcal{M}}$ want to compactify moduli spaces. Curves misleading too nice.

Two choices: Just study $\overline{\mathcal{M}}$, or study $\mathcal{M}$ if you can take a good "closure" of $\mathcal{M}$ without including bad components of $\overline{\mathcal{M}}$.

<u>Example</u>. The Tate curve

$$\begin{tikzcd}
\text{Smooth elliptic curve} \arrow[r, hook] \arrow[d, dash] & E_q \arrow[d, dash] & {} \arrow[l, hook'] \arrow[d, dash] \\
\operatorname{Spec} \mathbb{Z}[[q]][\tfrac{1}{q}] \arrow[r, hook] \arrow[d, dash] & \operatorname{Spec} \mathbb{Z}[[q]] & \operatorname{Spec} \mathbb{Z} \arrow[l, hook', "q=0"'] \arrow[d, dash] \\
\text{generic fiber} & & \text{special fiber}
\end{tikzcd}$$

$P = \operatorname{cone}(1, \mathbb{R})_{\mathbb{Z}} \subset \mathbb{R} \times \mathbb{R}.$



$P \rtimes \mathbb{N} =$ free monoid on generators x_n and q $\Big/$ $x_n + x_{n+2} = q + 2x_{n+1}$

This is a graded monoid with $\deg(x_i) = 1$, $\deg(q) = 0$.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, hook] & \operatorname{Proj}(\mathbb{Z}[P \rtimes \mathbb{N}]) \arrow[d, dash] \arrow[loop below] & {} \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[q^{\pm}] \arrow[r, hook] \arrow[d, dashed, dash] & \operatorname{Spec} \mathbb{Z}[q] & \operatorname{Spec} \mathbb{Z}. \arrow[l, hook', "q=0"'] \arrow[d, dashed, dash] \\
\text{generic fiber.} & & \text{Special fiber}
\end{tikzcd}$$

$\mathbb{Z}$ acts by shifting above picture up and down.

$$\rightsquigarrow \text{ quotient by } \mathbb{Z}\text{-action.}$$

$$\begin{tikzcd}
\text{elliptic curve} \arrow[r, hook] \arrow[d, dashed, dash] & E_q \arrow[d, dash] & {} \arrow[l, hook'] \arrow[d, dash] \\
\operatorname{Spec} \mathbb{Z}[[q]][q^{-1}] \arrow[r, dash] & \operatorname{Spec} \mathbb{Z}[[q]] & \operatorname{Spec} \mathbb{Z} \arrow[l, hook']
\end{tikzcd}$$

Q: How do you see toric structure on E_q?

A: log structure. (Fontaine, Illusie, Kato, ...)

X scheme.

$\mathcal{D}iv(X)$ groupoid of pairs (L, s)

L line bundle on X.

$s : L \longrightarrow \mathcal{O}_X$ ~~[illegible]~~.

$f : Y \to X$ gives

$$f^* : \mathcal{D}iv(X) \to \mathcal{D}iv(Y) \rightsquigarrow \text{Div stack on } X_{et}.$$

<u>Example</u> P a monoid

$$\begin{tikzcd} X = \operatorname{Spec} k[P] \arrow[r, squiggly] & P/F^{-1}(\mathrm{id}_{\mathcal{O}_X}) \arrow[r, dashed] \arrow[d, equal] & \mathrm{Div} \\ & \overline{\mathcal{M}} \arrow[ur, dashed] & \end{tikzcd}$$

Map

$P \to \mathcal{D}iv$ gives rise to

$$p \to (\mathcal{O}_X \xrightarrow{\cdot p} \mathcal{O}_X)$$

Support is on the toric boundary.

<u>Def.</u> A log structure on X is a pair $(\mathcal{M}, \alpha)$

$\mathcal{M}$ sheaf of monoids

$\alpha : \mathcal{M} \to \mathcal{O}_X$ s.t. $\alpha(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$.

Given $(\mathcal{M}, \alpha)$,

$$\begin{tikzcd} \overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}_X^* & \mathcal{M} \arrow[l, "\pi"'] \\ m \arrow[u, phantom, sloped, "\in"] \arrow[r] & \pi^{-1}(m) \end{tikzcd}$$

$\pi^{-1}(m)$ has an $\mathcal{O}_X^*$ action $\xleftrightarrow{\text{associated to}}$ line bundle L_m

using α, we get

$$L_m \xrightarrow{\alpha} \mathcal{O}_X.$$

$\rightsquigarrow$ Category of log structures.

$f : Y \to X$ gives $f^* : (\text{log structures on } X) \to (\text{log structures on } Y)$

<u>Def.</u> <u>Log scheme</u> is a pair (X, M_X) $\quad$ X scheme

M_X log structure

Category by morphisms

$$(Y, M_Y) \to (X, M_X) \quad \text{is} \quad f : Y \to X$$

$$f^b : f^* M_X \to M_Y .$$

Should think of this as generalization of pair (scheme, divisor).

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old picture becomes

$$\begin{tikzcd}
\text{Elliptic curve} \arrow[r, hook] & (E_q, M_{E_q}) \arrow[d, dash] & \propto \arrow[l, hook'] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}[[q]][\tfrac{1}{q}] \arrow[r, hook] & (\operatorname{Spec} \mathbb{Z}[[q]], M_q) & \operatorname{Spec} \mathbb{Z}. \arrow[l, hook']
\end{tikzcd}$$

Now consider abelian varieties $\mathcal{A}_{g,d}$. of dimension g, polarization degree d.

$$\mathcal{A}_{g,d}(S) = \{ (A, \lambda) \mid A/S \text{ abelian scheme of dim } g. \quad \lambda : A \to A^t \text{ polarization} \quad \operatorname{rank}(\ker(\lambda)) = d^2 .$$

Remark $d=1$.

Alexeev onstructed $\mathcal{A}_g \hookrightarrow \bar{\mathcal{A}}_g^{Alex.}$

How think of $\mathcal{A}_{g,d}$?

$\mathcal{T}_{g,d}$ classifies triples (A, P, L).

Think of P as an A torsor $f: P \to S$

L ample on P s.t. $f_* L$ rank d.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathcal{G}_{(P,L)} \arrow[r] \arrow[d, equal] & \ker(\lambda_L : A \to P^t = A^t) \arrow[r] & 1 \\
& & \{ \underset{A\text{-equiv}}{\sigma : P \simeq P} : \tau : \sigma^* L \xrightarrow{\sim} L \} & &
\end{tikzcd}$$

$\mathcal{A}_{g,d} \leftarrow \mathcal{T}_{g,d}$ rigidification by $\mathcal{G}$.

Standard construction for stacks.

Remark. If $d=1$, $\begin{tikzcd} \mathcal{T}_{g,d} \arrow[r, no head] & \mathcal{A}_{g,d} \arrow[l, bend right, "s"'] \end{tikzcd}$ in this case $\exists$ section s.

Degenerations: V a dvr. $K = \mathrm{frac}(V)$.

$(A_K, P_K, L_K) \in \mathcal{T}_{g,d}(K)$.

How can it degenerate?

$A_K \rightsquigarrow G$ semiabelian scheme $/V$.

G acts in log smooth category.

$$P_K \rightsquigarrow \overset{\circlearrowleft}{(P, M_P)} \longrightarrow (\mathrm{Spec}\, V, M_V)$$

log smooth, proper.

$L_K \rightsquigarrow L$ on P.

$$\Delta = (M_V, (P, M_P) \to (V, M_V), L, G).$$

$$\mathcal{G}_\Delta = \mathrm{Aut}_G\left(((P, M_P), L)/(V, M_V)\right)$$

If did degeneration right, get

$$1 - \mathbb{G}_m - \mathcal{G}_\Delta - \text{finite flat gp scheme of rank } d^2 \longrightarrow 1.$$

$$\begin{tikzcd}
\mathcal{T}_{g,d} \arrow[r, hook] \arrow[d] & \overline{\mathcal{T}}_{g,d} \arrow[d, "\text{rigidify by theta group.}"] \\
\mathcal{A}_{g,d} \arrow[r, hook] & \overline{\mathcal{A}}_{g,d}
\end{tikzcd}$$

$\overline{\mathcal{T}}_{g,d}$ classifies $(M_S, G, (P, M_P) \to (S, M_S), L)$.

<u>Thm.</u> $\overline{\mathcal{A}}_{g,d}$ proper over $\mathbb{Z}$, log smooth over $\mathbb{Z}[\frac{1}{d}]$.

$\mathcal{A}_{g,d} \hookrightarrow \overline{\mathcal{A}}_{g,d}$ dense open.

<u>Remarks</u>. 1. log smooth $\longleftrightarrow$ toric singularities.

2. for $d=1$, $\bar{\mathcal{A}}_g$ = normalization of MC in $\bar{A}_g^{Alex}$.

main component.

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classic work of Mumford: <u>Theta level structure</u>.

Fix integers $\delta = (d_1 \cdots d_g)$. $d = \prod d_i$ work over $\mathbb{Z}[\frac{1}{2d}]$.

$K(\delta) = \oplus \mathbb{Z}/(d_i)$.

$\hat{K}(\delta) = \mathrm{Hom}(K(\delta), \mathbb{G}_m)$.

$\tau \circlearrowright \mathcal{G}(\delta) \quad \mathbb{G}_m \times K(\delta) \times \hat{K}(\delta)$ mult. defined by

involution

$$(\alpha, x, \ell)\cdot(\alpha', x', \ell') = (\alpha\alpha'\ell(x'), x+x', \ell+\ell').$$

$\mathcal{M}_{g,\delta}(S)$ objects are tuples $(A, L, \sigma, \varepsilon : e^* L \simeq \mathcal{O}_S)$.

- A/S abelian scheme.
- L rel. ample on A, symmetric.
- $\sigma : \mathcal{G}(\delta)_S \simeq \mathcal{G}_{(A,\rho,L)}$ identity on $\mathbb{G}_m$.

One thing Mumford showed is if

$8 \mid d_i \ \forall i$, then $\mathcal{M}^{tot}_{g,\delta}$ is a quasi projective variety.

Reason: $\exists!$ irred. rep V_δ of $\mathcal{G}(\delta)$ of rank d on which $\mathbb{G}_m$ acts by multiplication.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "e"'] & \mathbb{P}(f_* L) \arrow[r, phantom, "\simeq"] & \mathbb{P} V_{\delta, S} \\ S \arrow[urr, dashed, leftrightarrow, "\text{canonical iso}"'] & & \end{tikzcd}$$

induces a map $\mathcal{M}^{tot}_{g,\delta} \hookrightarrow \mathbb{P} V_\delta$, which Mumford shows is an embedding.

This is roughly embedding by theta functions.

Q: How to resolve singularities of the closure?

$$\begin{tikzcd} \overline{\Theta}^{tot}_{g,\delta} \arrow[r] & \overline{\mathcal{M}}^{tot}_{g,\delta} \end{tikzcd}$$

↑ log smooth.

$\mathcal{M}_{g,\delta}$
$=$
rigidification of Fix(ρ).

canonical iso interchanging torsor structure.

$$\begin{tikzcd} P \arrow[rr, "\rho", "\sim"'] \arrow[dr] & & P^i \arrow[dl] \\ & \Sigma_{g,\delta} = \mathrm{Isom}(\mathcal{G}, \mathcal{G}(\delta)) \arrow[d] & \\ & T_{g,\delta} & \end{tikzcd}$$