BURT TOTARO: ALGEBRAIC SURFACES AND HYPERBOLIC GEOMETRY

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Let X be a smooth complex projective variety. There are three interesting types of varieties for study: fano ($-K_X$ ample), Calabi-Yau ($K_X \equiv 0$), and general type (ample K_X). Here, for $\dim_{\mathbb{C}} X = n$, the canonical bundle K_X is Ω^n_X. We write $-K_X$ for K^*_X.

Example. For $\dim_X = 1$, we have $\mathbb{CP}^1 = S^2$ (genus 0), elliptic curves (genus 1), and general type curves (genus ≥ 2). For surfaces in $\mathbb{P}^3$, we have those of degree $\leq 3, 4$, and ≥ 5 respectively. These three classes are equivalent to the existence of a Kähler metric with Ricci curvature $> 0, 0, < 0$ respectively. We further have boundedness for smooth Fano n-folds (Kollar, Miyaoka, Mori, Campana).

Example. A smooth Fano surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ at ≤ 8 points.

Definition 1. *A line bundle L on X is ample if $\exists n > 0$ s.t. $L^{\otimes n}$ has enough global sections to give a projective embedding $X \to \mathbb{P}^N$, wher $N = \dim_{\mathbb{C}} H^0(X, L^{\otimes N}) - 1$.*

Example. For X a curve, L is ample iff $\deg L < 0$, and we write $L \cdot X = \deg L|_X \in \mathbb{Z}$.

Numerical criterion of ampleness: an $\mathbb{R}$-divisor on X is a finite sum $D = \sum a_i D_i, a_i \in \mathbb{R}, D_i$ an irred divisor in X.

Definition 2. *$N^1(X)$ is the real vector space of $\mathbb{R}$ divisors modulo numerical equivalence: $D_1 \equiv D_2 \Leftrightarrow D_1 \cdot C = D_2 \cdot C$ for all curves C in X (for me, a curve is irreducible).*

Over $\mathbb{C}$, $N^1(X) \subset H^2(X, \mathbb{R})$. The dual vector space is $N_1(X) = \{a_i C_i : a_i \in \mathbb{R}, C_i$ curves in $X\}$ modulo numerical equivalence. Over $\mathbb{C}, N_1 \subset H_2(X, \mathbb{R})$.

Definition 3. *The closed cone of curves $\overline{NE}(X)$ is the closed conves cone in $N_1(X)$ spanned by curves on X.*

An $\mathbb{R}$-divisor D is nef if $D \cdot C \geq 0$ for all curves C in X (likewise for line bundles). Thus $Nef(X) \subset N^1$ is a closed convex cone, the dual cone to $\overline{NE}(X) \subset N_1(X)$.

Theorem 1 (Kleiman)**.** *A line bundle L on X is ample $\Leftrightarrow [L]$ is in the interior of $Nef(X) \subset N^1(X)$.*

This is a numerical characterization of ampleness. It shows that we know $\mathrm{Amp}(X) \subset N^1(X)$ if we know the cone of curves $NE(X) \subset N_1(X)$.

Theorem 2 (Cone theorem: Mori, Shokurov, Kawamata, Reid, Kollar)**.** *Let X be a smooth projective variety, and write $K_X^{<0} = \{u \in N_1(X) : K_X \cdot u < 0\}$ Then every extremal ray of $\overline{NE}(X)_{\cap K_X^{<0}}$ is isolated, rational, spanned by a curve, and can be contracted.*

Corollary 1. *For a Fano variety X, the cone of curves (and thus the dual cone* Nef*) is rational polyhedral.*

Proof. $-K_X$ is ample, so K_X is negative on all of $\overline{NE}(X) \smallsetminus \{0\}$. □

Just beyond Fano varieties, these cones need not be $\mathbb{Q}$-polyhedral.

Example. Let X be the blowup of $\mathbb{P}^2$ at n very general points. For $n \leq 8$, X is fano, so $\overline{NE}(X)$ is $\mathbb{Q}$-polyhedral: for $3 \leq n \leq 8$, $\overline{NE}(X)$ is the convex cone spanned by the finitely many (-1)-curves in X, $C \cong \mathbb{P}^1, C^2 = -1$. But for $n \geq 9$, X is not Fano: $(-K_X)^2 = 9 - n$. For $p_1, \ldots, p_n$ very general, X contains infinitely many (-1)-curves. Every curve C with $C^2 < 0$ on aa surface spans an isolated extremal ray of $\overline{NE}(X)$, so $\overline{NE}(X)$ is not $\mathbb{Q}$-polyhedral. (The positive side of $\overline{NE}(X)$ is unknown.)

Calabi-Yau varieties ($K_X \equiv 0$) are also just beyond Fano varieties. Again the cone of curves of a Calabi-Yau manifold need not be $\mathbb{Q}$-polyhedral.

Example. Let X be an abelian surface, $X \cong \mathbb{C}^2/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^4$, X projective. Then $\overline{NE}(X) = Nef(X)$ is a round cone, $= \{x \in N^1(X) : x^2 \geq 0$ and $H \cdot x \geq 0\}$. This is not $\mathbb{Q}$-polyhedral if X has Picard number $\rho(X) := \dim_{\mathbb{R}} N^1(X)$ at least 3 (and sometimes when $\rho = 2$). For a K3 surface, $\overline{NE}(X)$ may be round, or may be the closed cone spanned by the (-2)-curves in X (Kovacs). There may be finitely or infinitely many (-2)-curves.

But there is a good substitute for the cone theorem for Calabi-Yau varieties (proved in dimension 2 and conjectured in general).

Theorem 3 (Sterk, Looijenga, Namikawa)**.** *Let X be a smooth projective Calabi-Yau surface ($K_X \equiv 0$). Then the action of* $\mathrm{Aut}(X)$ *on the nef cone has a $\mathbb{Q}$-polyhedral fundamental domain.*

Remark. For any variety X, if $\mathrm{Nef}(X)$ is $\mathbb{Q}$-polyhedral, then

$$\mathrm{Aut}^*(X) := \mathrm{Im}\ (\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))) \quad (1)$$

is finite (easy). The theorem implies the converse for Calabi-Yau surfaces.

The Sterk theorem should generalize to Calabi-Yaus of arbitrary dimension (the *Morrison-Kawamata cone conjecture*). But in dimension 2, we can visualize it better using hyperbolic geometry. Indeed, let X be any smooth projective surface. The intersection form on $N^1(X)$ always has signature $(1, n)$ for some n by the Hodge index theorem. So $\{x \in N^1(X) : x^2 > 0\}$ is the standard round cone. And we can identify the positive cone $\{x^2 > 0, H \cdot x > 0\}$ modulo $\mathbb{R}^{>0}$ with hyperbolic n-space. That is, the Lorentzian metric on $N^1(X) = \mathbb{R}^{1,n}$ restricted to the quadric $\{x^2 = 1\}$ is a Riemannian maetric with curvature -1.

For any algebraic surface X, $\mathrm{Aut}(X)$ preserves the intersection form on $N^1(X)$, so $\mathrm{Aut}^*(X) = \mathrm{Im}\ (\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)))$ is always a group of isometries of hyperbolic n-spacee, where $n = \rho(X) - 1$.

Example. For X an abelian surface with $\rho(X) = 3$, the cone $\mathrm{Nef}(X)$ is round, so $\mathrm{Aut}^*(X)$ must be infinite. For instance, let $X = E \times E$ with E a non-CM elliptic curve. Then $\rho(X) = 3$, spanned by $E \times \mathrm{pt}, \mathrm{pt} \times E$ and $\Delta_E \subset E \times E$. So $\mathrm{Aut}^*(X)$ must be infinite, and in fact $\mathrm{SL}(2, \mathbb{Z})$. The Sterk theorem says that the action on this on the hyperbolic plane has a $\mathbb{Q}$-polyhedral fundamental domain.

For a K3 surface X the cone $\mathrm{Nef}(X)$ may or may not be the whole positive cone. In general, the nef cone mod scalars is a *convex subset* of hyperbolic space. A finite polytope in hyperbolic space (even if some vertices are at infinity) has finite volume. So the Sterk theorem implies that $\mathrm{Aut}(X)$ acts with finite covolume on the convex set $\mathrm{Nef}(X)/\mathbb{R}^{>0}$ in hyperbolic space. The proof of Sterk relies on the Torelli theorem of K3 surfaces of Piatetski-Shapiro and Shafarevich. That is, any isomorphism of Hodge structures between the K3s is realized by an isomorphism of K3s if it maps the nef cone into the nef cone. In particular, this lets us construct automorphisms of a K3 X: up to finite index, every element of $O(\mathrm{Pic}X) \cong \mathbb{Z}^p$ that preserves the cone $\mathrm{Nef}(X)$ is realized by an automorphsim of X. Moreover, $\mathrm{Nef}(X)/\mathbb{R}^{>0} \subset H_n$ (hyperbolic space) is a very special convex set: it is a Weyl chamber for a discrete reflection group N acting on H_n. $N =$ the group ggenerated by the reflections in vectors $x \in \mathrm{Pic}X$ with $x^2 = -2$, i.e. the group generated by reflections in (-2)-curves. By the first description, N is the normal subgroup of $O(\mathrm{Pic}X)$, and in fact $\mathrm{Aut}(X) \rtimes N$. By general results on arithmetic groups, $O(\mathrm{Pic}X)$ acts on the positive cone with a $\mathbb{Q}$-polyhedral fundamental domain (not unique). Therefore, $\mathrm{Aut}(X)$ acts on the nef coone with the same $\mathbb{Q}$-polyhedral fundamental domain (up to finite index).

Definition 4. *A* pair (X, Δ) *is normal $\mathbb{Q}$-factorial projective variety X with an effective $\mathbb{R}$-divisor Δ on X.*

We think of $K_X + \Delta$ as the canonical bundle of the pair.

Definition 5. *A pair (K, Δ) is klt (Kawamata log terminal) if the following holds: let $\pi : \tilde{X} \to X$ be a resolution of singularities such that* $\mathrm{Exc}(\pi) \cup \pi^{-1}(\Delta)$ *is a divisor with simple normal crossings. Define a divisor $\tilde{\Delta}$ on $\tilde{X}$ by $K_{\tilde{X}} + \tilde{\Delta} = \pi^*(K_X + \Delta)$. Then $\tilde{\Delta}$ has coefficients* < 1.

Example. A surface $X = (X, 0)$ is klt iff it has only quotient singularities. For X smooth and Δ a snc divisor (and some coefficients), the pair (X, Δ) is klt iff Δ has coefficients < 1.

All the main results of minimal model theory such as the cone theorem generalize from smooth varieties to klt pairs. For example, the Fano case of the cone theorem becomees:

Theorem 4. *Let (X, Δ) be a klt Fano pair ($-(K_X+\Delta)$ is ample). Then $\overline{NE}(X)$ (and hence the dual cone $\mathrm{Nef}(X)$) is $\mathbb{Q}$-polyhedral.*

Example. Let X be the blowup of P^2 at a number of points on a smooth conic. Then $\exists\Delta$ s.t. (X, Δ) is a klt Fano pair (cf. Castravet-Tevelev; BCHM; Mukai; Testa, Varilly-Alvarado and Velasco).

It is therefore natural to extend theMorrison-Kawamata cone conjecture from Calabi-Yau varieties to Calabi-Yau pairs ($K_X + \Delta \equiv 0$). The conjecture is reasonable, since we can prove it in dimension 2.

Theorem 5. *Let (X, Δ) be a klt CY surface. Then $\mathrm{Aut}(X, \Delta)$ (and also $\mathrm{Aut}(X)$) acts with a $\mathbb{Q}$-polyhedral fundamental domain on the cone $\mathrm{Nef}(X) \subset N^1(X)$.*

The conclusion is false for surfaces in general, even for some smooth rational surfaces.

Example. let X be the blowup of $\mathbb{P}^2$ at 9 very general points. Then $\mathrm{Nef}(X)$ is not $\mathbb{Q}$-polyhedral, since X contains infinitely many (-1)-curves. But $\mathrm{Aut}(X) = 1$, so the conclusion fails forr X¡ Moreover, let Δ be the unique smooth cubic in $\mathbb{P}^2$ through the 9 points, with coefficient 1. Then $-K_X \equiv \Delta$, so (X, Δ) is a *log canonical* Calabi-Yau pair. The theorem therefore fails again.

Here is an example where the theorem applies:

Example. Let X be the blowup of $\mathbb{P}^2$ at 9 points which are the intersection of two cubic curves. Then taking linear combinations of the two curves gives a $\mathbb{P}^1$ family of elliptic curves through the nine points, which become disjoint on the blowup X, so we have an elliptic fibration $X \to \mathbb{P}^1$. This morphism is given by the linear system $|-K_X|$, and we see that the (-1)-curves in X are exactly the sections of the elliptic fibration $X \to \mathbb{P}^1$. In most cases, the Mordell-Weil group (group of sections) of $X \to \mathbb{P}^1$ is $\cong \mathbb{Z}^8$. So X contains infinitely many (-1)-curves, and so $\mathrm{Nef}(X)$ is not $\mathbb{Q}$-polyhedral. But $\mathrm{Aut}(X)$ acts transitively on the set of (-1)-curves, here. That leads to the proof of the theorem in this case (there is a Δ with $K_X + \Delta$ CY).

Corollary 2. *Let (X, Δ) be a klt CY surface. Then X has only finitely many contractions $X \to Y$ up to automorphisms of X. Related to that:* $\mathrm{Aut}(X)$ *has finitely many orbits on the set of curves in X with negative self-intersection.*

In higher dimensions, the cone conjecture also predicts that a klt CY pair (X, Δ) has only finitely many small $\mathbb{Q}$-factorial modifications (SQMs) up to automorphisms of X (There can be infinitely many SQMs if we don't divide out by ${}^{p}\mathrm{Aut}(X)$, the group of birational automorphisms of X which are isomorphisms in codimension 1. Kawamata proved a relativeversion of the cone conjecture for X/S with X a 3-fold and $X \to S$ a K3 fibration or elliptic fibration. Here X can have infinitely miny minimal models ($\implies$ SQMs) over S, but finitely many modles $Ps\mathrm{Aut}(X)$.

Proof of theorem. Let (X, Δ) be a klt CY surface. We can reduce to the case X smooth. We know this for $\Delta = 0$, so assume otherwise. Then X has Kodaira dimension $\kappa(X) = \kappa(X, K_X) = -\infty$. With one exception, Nikalin showed that our assumpptions imply that X is rational, so assume that from now on. We have three main cases foor the proof, depeneding on whether the Iitaka dimension $\kappa(X, -K_X)$ is $0, 1, 2$.

Case 2: that is, K_X is big. Then $\exists \Gamma$ s.t. (X, Γ) is klt Fano. Therefore, Nef(X) is $\mathbb{Q}$-polyhedral, and therefore $\mathrm{Aut}^*(X)$ is finite. So the theorem is true in a simple way. For generally, for (X, Γ) klt Fano of any dimension, the Cox ring of X is finitely generated by BCHM.

Case 1: in this case, $-K_X$ determines an elliptic fibration of X, not necessarily minimal, $X \to P^1$. Here $\mathrm{Aut}^*(X)$ is the Mordell-Weil group of $X \to \mathbb{P}^1 \cong \mathbb{Z}^n$ for some n. The (-1)-cuurves on X are certain multisections of $X \to \mathbb{P}^1$. We show that $\mathrm{Aut}(X)$ has only finitely many orbits on the (-1) curves of X, which leads to the result.

Case 0: which is the hardest case. Here, $\mathrm{Aut}^*(X)$ can be a fairly general group acting on hyperbolic space, and $-K_X = \Delta$ where the intersection pairing on the curves in Δ is negative definite. We caan contract all the curves in Δ. Let I be the "global index" of Y, the least positive integerr with $I \cdot K_Y$ Cartier and ~ 0. Then $Y = M/(\mathbb{Z}/I)$ with M a CY surface with ADE singularities. The minimal resolution of M is a smooth CY surface. This leads to the proof of the theorem for M and then for Y.

Finally, we have to go from Y to its resolution X. Here, $\mathrm{Nef}(X)$ is more complex than $\mathrm{Nef}(Y)$: X typically contains infinitely many (-1)-curves, whereas Y has none (since $K_Y \equiv 0$). Nonetheless, since we know "'how big"' $\mathrm{Aut}(Y)$ is (up to finite index), we can show that $\mathrm{Aut}(X, \Delta) = \mathrm{Aut}(Y)$ has finitely many orbits on the set of (-1) curves in X. This leads to the proof for (X, Δ). □

Example. We consider the case of a smooth rational surface with an infinite, highly nonabelian (discrete) automorphism group. Also, a singular rational surface whose nef cone is round of dimension 4. Let X be the blowup of $\mathbb{P}^2$ at 12 points: $[1, \zeta^i, \zeta^j]$ for $i, j \in \mathbb{Z}/3\mathbb{Z}$ and $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. Here ζ is a cube root of 1 (dual to the Hesse configuration). There are 9 lines through quadruples of the 12 points in $\mathbb{P}^2$.

Contract the 4 (-3)-curves L_i on X. This gives a rational surface Y with 9 singular points (of type $\frac{1}{3}(1,1)$) and $\rho(Y) = 4$). $Y : -K_Y \equiv 0$, Y is klt CY surface (which is rational). Since $3K_Y \sim 0$, $Y \cong M/(\mathbb{Z}/3\mathbb{Z})$ with M a CY surface with ADE singularities. In fact, M is smooth, $M \cong E \times E$ where $E \cong \mathbb{C}/\mathbb{Z}[\zeta]$ and $\mathbb{Z}/3\mathbb{Z}$ acts on $E \times E$ as multiplication by (ζ, ζ). Therefore, $\mathrm{GL}(2, \mathbb{Z}[\zeta])$ acts on M. This passes to an action on Y and hence on its minimal resolution X ($= \mathbb{P}^2$ blown up at 12 points). Here $\mathrm{Nef}(Y) = \mathrm{Nef}(M)$ is a round cone in $\mathbb{R}^4$, so this group acts on the hyperbolic 3-space.