ALINA MARIAN: LIE ALGEBRA ACTIONS ON THE COHOMOLOGY OF HYPERQUOT SCHEMES

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Let X be a quasi-projective variety, and let $\mathcal{M}_X(c)$ be a moduli space of sheaves on X with topological type c. If we look at the union of all such moduli spaces $\sqcup_c \mathcal{M}_X(c)$ and look at two topological types c, c', we obtain $Z_{c,c'} \subset \mathcal{M}_X(c) \times \mathcal{M}_X(c')$ which generate an algebra which acts on $\mathbb{H} = \bigoplus_c H^*(\mathcal{M}_X(c))$.

Moreover, the class $[Z_{c,c'}]$ gives a map

$$H^*(\mathcal{M}_X(c)) \to H^*(\mathcal{M}_X(c')), \alpha \mapsto p_{2*}(p_1^*\alpha \cdot [Z_{c,c'}]) \tag{1}$$

On $\bigoplus_{n\geq 0} H^*(S^{[n]})$, there is an action of an infinite dimensional Heisenberg algebra

$$[P_m, P_n] = m\delta_{m+n,0}K, [P_m, K] = 0 \tag{2}$$

Fix N, and consider $\bigsqcup_{0\leq r\leq n} G(r,n)$.

Theorem 1 (Nakajima, Ginzburg)**.** *There is an $\mathfrak{sl}_2$-action on $\bigoplus_{r=0}^N H_*(G(r,N))$ such that span of the $[G(r,N)]$ over all r give the irreducible highest weight N $\mathfrak{sl}_2$-module.*

Explicitly, exhibiting $\mathfrak{sl}_2$ via generators $e, f, h, [e,f] = h, [h,e] = 2e, [h,f] = 2f$ and looking at

$$\begin{array}{ccc} C_r = \{V \subset V'\} \subset G(r,N) \times G(r+1,N) & & \\ \downarrow & \searrow & \\ G(r,N) & & G(r+1,N) \end{array} \tag{3}$$

we would like to set $e : \alpha \mapsto p_{2*}(p_1^*\alpha \cdot [C_r])$. This, however, is too naive, so we instead consider

$$\mathcal{N}^\vee_{C_r/T^*G(r,N)\times T^*G(r+1,N)} \subset T^*(G(r,N) \times T^*(G(r+1,N))) \tag{4}$$

We get a well-defned e, f s.t.. $[e,f] = (-1)^N \oplus (N-2r)\text{id}_{\ H_*(G(r,N))}$ and

$$\mathbb{H}_* = \bigoplus_{r=0}^N H_*(G(r,N)) \cong (\mathbb{C}^2)^{\otimes N} \tag{5}$$

This was used by Nakajima for flag varieties, and Ginzburg for quiver varieties.

We look at the case of complete flag varieties $\mathrm{Fl}_n = SL(n,\mathbb{C})/$upper-triangular group $= \{V_1 \subset \cdots \subset V_{n-1} \subset O^n\}$, $H^2(\mathrm{Fl}_n) \cong \mathbb{Z}^{n-1}$ spanned by first Chern classes of universal bundles. We obtain a smooth quasi-projective space

$$\mathrm{Mor}_{\underline{d}}(\mathbb{P}^1, \mathrm{Fl}_n) = \{A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_{n-1} \hookrightarrow O^n \to B_1 \to \cdots\} \tag{6}$$

(all the maps over $\mathbb{P}^1$) of dimension $\frac{n(n-1)}{2}+2|\underline{d}|$, $|\underline{d}| = d_1+\cdots+d_n$. Compactifying to $Q_{\underline{d}}(\mathrm{Fl}_n)$ we obtain a parameter space for $A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_{n-1} \hookrightarrow O^n \to B_1 \to \cdots$ which is a fine moduli space. There is a universal structure $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{A}_{n-1} \hookrightarrow O^n \to \mathcal{B}_1 \to \cdots$ over $Q_{\underline{d}} \times \mathbb{P}^1$. Note that the $\mathcal{A}_i$'s are all vector bundles, though the $\mathcal{B}_i$'s are not.

Remark. The (Künneth components of the) Chern classes of $\mathcal{A}_i$ generate $H^*(Q_{\underline{d}}, \mathbb{Q})$ multiplicatively.

We can calculate Betti numbers via torus actions using localization, splitting $O^n = \bigoplus_{r=1}^n L_i \otimes O$ as eigenlines and getting a filtration $M_1 \hookrightarrow M_{21} \oplus M_{22} \hookrightarrow \cdots \hookrightarrow O^n$.

Fact (Linda Chen): we obtain a multiplicative decomposition

$$\begin{aligned} P(t,z) &= \sum_{\underline{d},m\geq 0} b_{2m}(Q_{\underline{d}}) z^m t_1^{d_1} \cdots t_{n-1}^{d_{n-1}} \\ &= \prod_{1\leq i\leq j\leq n-1} \frac{1}{1-t_i\cdots t_j z^{j-i}} \frac{1}{1-t_i\cdots t_j z^{j-i+2}} \end{aligned} \tag{7}$$

Look at

$$\prod_{1\leq i\leq j\leq n-1} \frac{1}{1-t_i\cdots t_j} \tag{8}$$

which is the character of a Verma module. Setting $g = \underbrace{h \oplus n^-}_{b} \oplus n$ where $h|0\rangle = \lambda(h)|0\rangle, n^-|0\rangle = 0$ and n acts freely, this representation is the Verma module $V_\lambda = \mathcal{U}(g) \otimes_{\mathcal{U}(b)} |0 > \cong \mathcal{U}(n)$. The character of V is

$$\sum_{\mu \text{ weight}} \dim V_\mu \cdot t^\mu = \sum P(\mu) t^\mu = \prod_\alpha \frac{1}{1-t^\alpha} \tag{9}$$

where $P(\mu)$ is the number of ways that one can right a weight as a sum of positive roots, and α runs through the positive roots. In the case of $\mathfrak{sl}_n$, we obtain the above expression. Now define $C_{\underline{d},i} \subset Q_{\underline{d}} \times Q_{\underline{d}+i}$ as the set of $A_i \hookrightarrow O^n, A_i^1 \subset O^n$

s.t.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbb{C}_p & & & & & & \\
 & & & & & & \uparrow & & & & & & \\
A_1 & \longrightarrow & \cdots & \longrightarrow & A_{i-1} & \longrightarrow & A_i & \longrightarrow & A_{i+1} & \longrightarrow & \cdots & \longrightarrow & O^n \\
\uparrow{\scriptstyle =} & & & & \uparrow{\scriptstyle =} & & \hookuparrow & & \uparrow{\scriptstyle =} & & & & \\
A'_1 & \longrightarrow & \cdots & \longrightarrow & A'_{i-1} & \longrightarrow & A'_{i-1} & \longrightarrow & A'_{i+1} & \longrightarrow & \cdots & \longrightarrow & O^n
\end{array}
\tag{10}
$$

This induces

$$
\begin{aligned}
e_{\underline{d},i} &: H^*(Q_{\underline{d}}) \to H^{*+2}(Q_{\underline{d}+i}) \\
f_{\underline{d},i} &: H^*(Q_{\underline{d}}) \to H^{*-2}(Q_{\underline{d}-i})
\end{aligned}
\tag{11}
$$

and induced e_i, f_i.

Theorem 2 (Finkelberg, Kuznetsov)**.** $[e_i, f_i] = \bigoplus_{\underline{d}}(2d_i - d_{i-1} - d_{i+1} + 2)\mathrm{id}_{\ H^*(Q_{\underline{d}})}$ *where* $e_i, f_i, h_i = 2d_i - d_{i-1} - d_{i+1} + 2$ *satisfy the relations for* $\mathfrak{sl}_n$

$$
[e_i, f_j] = h_i, [h_i, e_i] = c_{ij}e_j, [h_i, f_j] = -c_{ij}f_j
\tag{12}
$$

where $c_{ij} = -2$ *if* $i = j$, -1 *if* $|i - j| = 1$, *and* 0 *otherwise.*

Now consider $\mathcal{R}_{\underline{d}} \subset Q_{\underline{d}}$ the space of flags of sheaves A_i s.t. $A_i|_\infty = V_i$, $V_1 \subset V_2 \subset \cdots \subset V \otimes O = O^n$. This is quasiprojective of dimension $2|d|$, and $\bigoplus_{\underline{d}} H^*_{T\times\mathbb{C}^*}(\mathcal{R}_d) \otimes F$ carries an $\mathfrak{sl}_n$-action.

Equivalently, we can see two $\mathfrak{sl}_n$-actions (Frenkel, Finkelberg, et al 2008).

Conjecture 1. *There are operators* $P_\alpha, Q_\alpha, \alpha \in \Phi^\perp$ *(*$\Phi^\perp$ *the set of positive roots of* $\mathfrak{sl}_n$*) all commuting and acting freely on the space of vacuum* $H^*(\mathrm{Fl}_n)$*. For* α *a simple root,* P_α *maps* $H_*(Q_{\underline{d}}) \to H_{*+0}(Q_{\underline{d}})$ *and* Q_α *maps* $H_*(Q_{\underline{d}}) \to H_{*+4}(Q_{\underline{d}})$*. Looking at* $Q_{\underline{d}} \times Q_{\underline{d}+[i,j]}$*, where* $\underline{d} + [i,j] = d_1 \ldots d_i + 1 \ldots d_j + 1 \ldots$*, we have subset* $C_{\underline{d},[i,j]}$ *parameterizing*

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbb{C}_p & & & & \mathbb{C}_p & & \\
 & & & & & & \uparrow & & & & \uparrow & & \\
A_1 & \longrightarrow & \cdots & \longrightarrow & A_{i-1} & \longrightarrow & A_i & \longrightarrow & \cdots & \longrightarrow & A_j & \longrightarrow \cdots \longrightarrow & O^n \\
\uparrow{\scriptstyle =} & & & & \uparrow{\scriptstyle =} & & \hookuparrow & & & & \uparrow{\scriptstyle =} & & \\
A'_1 & \longrightarrow & \cdots & \longrightarrow & A'_{i-1} & \longrightarrow & A'_i & \longrightarrow & \cdots & \longrightarrow & A'_j & \longrightarrow \cdots \longrightarrow & O^n
\end{array}
\tag{13}
$$

The P_α*'s induced by these* $C_{\underline{d},[i,j]}$ *commute.*