KIERAN O'GRADY: FOUR DIMENSIONAL ANALOGUES OF K3 SURFACES

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Definition 1. *A K3 surface is a compact Kähler surface such that* $B_1(X) = 0, K_X \cong \mathcal{O}_X$.

For instance, the standard quartic in $\mathbb{P}^3$. The generic K3 is not projective but does deform to one. We recall two key results on K3 surfaces.

Theorem 1 (Kodaira)**.** *Any two K3 surfaces are deformation equivalent, i.e. there is one deformation class of K3s, i.e. any K3 can be deformed to the standard quartic.*

Theorem 2 (Shafarevich and P.S., Burns-Rappaport,...)**.** *Global Torelli holds for K3 surfaces:* X, Y *are isomorphic K3 surfaces iff* $\exists : H^2(X, \mathbb{C}) \xrightarrow{\sim} H^2(Y, \mathbb{C})$ *is an integral isometry which respects the Hodge isomorphism, i.e. it maps* $H^{2,0}(X) \to H^{2,0}(Y)$ *isomorphically.*

We can construct higher dimensional analogues of these surfaces, called *hyperkähler manifolds*.

Definition 2. *A* hyperkähler manifold X^{2n} *is a Kähler manifold s.t.* $\pi_1(X) = \{1\}, H^0(\Omega^2) = \mathbb{C}\sigma$ *for some nondegenerate (symplectic) form* σ*. In particular,* $\wedge^n \sigma \in H^0(K_X)$ *is nowhere zero so* $K_X \cong \mathcal{O}_X$.

These are one of the building blocks compact Kähler manifolds with $c_1 = 0$, the others being the complex tori $\mathbb{C}^m$/lattices and *proper Calabi-Yau manifolds* which have $h^0(\Omega^p_X) = 0$ for $0 < p < n$. The hyperkähler ones are those with interesting holonomy (due to Yau's proof of the Calabi conjecture).

Example. For dim $X = 2$, the only hyperkähler manifolds are K3 surfaces. In dimension 4, blowing up the symmetric square of a K3 on the diagonal (called the Hilbert square $S^{[2]}$) gives one, as the symplectic form on the K3 lifts. Moreover, for $V \subset \mathbb{P}^5$ a cubic hypersurface, $X(V) = \{\ell \subset V | \ell \text{ a line}\}$, $X(V)$ is hyperkähler deformation equivalent to the above example if V is smooth, with the associated K3 being $S = \{p \in \ell \subset V_0\}$ for $V_0 = V(f_2 + f_3)$ the intersection of a quadric and a cubic in $\mathbb{P}^4$ and p the ordinary double point on V_0. The map $S^{[2]} \to X(V_0)$ is given by $(\ell_1, \ell_2) \mapsto \ell$ for ℓ the third line on the plane spanned by ℓ_1, ℓ_2.

Remark. The number of moduli of $S^{[2]}$ is 20 (arbitrary K3) or 19 (projective K3), as $S^{[2]}$ is projective if S is. However, the number of moduli of $X(V)$ is 20 (dimension of the space of cubic polynomials), so there are more HK manifolds than K3s and there is a locally complete family of projective HK manifolds.

Questions:

(1) How many deformational classes of HK manifolds are there? At least two in each (even) dimension ≥ 4.
(2) Does global Torelli hold? No in general, but for pairs of deformation equivalent HK manifolds, it is conjectured that they will be birational if ϕ satisfies the same condition.

We can think of tori as being governed by H^1 (curve-like), proper CYs by H^n, and HKs by H^2 (surface-like).

Definition 3. *The* Bogomolov-Beauville-Fujiki form *is a nondegenerate symmetric form q on $H^2(X,\mathbb{Z})$ defined by*

$$q(\beta_1,\beta_2)=\int_X \omega^{2n-2}\wedge\beta_2\wedge\alpha_2-\frac{2(n-1)\int_X\omega^{2n-1}\wedge\beta_1\cdot\int_X\omega^{2n-1}\wedge\beta_2}{(2n-1)^2\int_X\omega^{2n}} \tag{1}$$

Up to a constant Q, q is uniquely defined by $\int_X \beta^{2n}=c_X q(\beta,\beta)^n$ for $c_X\in\mathbb{Q}_+$. For $\sigma\in H^0(\Omega_X^2)$, $q_X(\sigma)=0, q_X(\sigma+\overline{\sigma})>0$. We then obtain a smooth period map $\mathrm{Def}(X)\to\{[\sigma]\in\mathbb{P}(H^2(X))|q_X(\sigma)=0,q_X(\sigma+\overline{\sigma})>0\}$ (Local) infinitesimal Torelli says that this map is a local isomorphism.

Strategy for proof: show that if X is numerically a $(\mathrm{K3})^{[2]}$, i.e. $q_X\cong q_{(\mathrm{K3}^{[2]})}$ and $c_X=c_{(\mathrm{K3})^{[2]}}=3$, then X is a deformation of a $(\mathrm{K3}^{[2]})$, and then that "Naive" Global Torelli holds for $(\mathrm{K3}^{[2]})$.

We consider the EPW (Eisenbud-Popescu-Walter) sextic Y: for $\bigwedge^3\mathbb{C}^6\times\bigwedge^3\mathbb{C}^6\to\mathbb{C}$ a symplectic form, $\alpha,\beta\mapsto\mathrm{vol}(\alpha\wedge\beta)$,

$$LG=LG(\bigwedge^3\mathbb{C}^6)=\{A\subset\bigwedge^3\mathbb{C}^6\text{ Lagrangian}\} \tag{2}$$

For $[v]\in\mathbb{P}^5=\mathbb{P}(\mathbb{C}^6)$, set

$$LG\ni F_v=\{\alpha\in\bigwedge^3\mathbb{C}^6|\sigma\wedge\alpha=0\} \tag{3}$$

, and for $A\in LG$, set

$$Y_A=\{[v]\in\mathbb{P}^5|A\cap F_v\neq\{0\}\} \tag{4}$$

This is the special sextic Y, unless A is pathological and $Y_A\cong\mathbb{P}^5$. Another description of Y: consider $\mathrm{Gr}=\mathrm{Gr}(1,\mathbb{P}^4)\subset\mathbb{P}^9$ via the Plčker embedding, and let $Z_A=\mathrm{Gr}\cap Q_A$ for $Q_A\subset\mathbb{P}^9$ a quadric; then $Y_A\subset|\mathcal{I}_{Z_A}(2)|$ is the space of singular quadrics Q minus $\mathcal{I}_{\mathrm{Gr}}(2)$. There is a natural double cover $f_A:X_A\to Y_A$ for A generic ramified only over the singular locus of Y_A (which is a smooth surface).

Theorem 3 (O'Grady)**.** *If A is generic, X_A is a deformation of $(K3)^{[2]}$, and $H_A = f_A^*\mathcal{O}_{Y_A}(1)$ satisfies $q(H_A) = 2$ and gives a locally complete family $\{(X_A, H_A)\}_{A \in LG}$. Moreover, if Z is numerically $(K3)^{[2]}$, we can deform it to X equipped with ample H s.t. $q_X(H) = 2$, and for $X \dashrightarrow |H|^\vee \cong \mathbb{P}^5$, one of the following holds:*

- *(X, H) is a deformation of (X_A, H_A) for $A \in LG$, or*
- *f is birational onto its image.*

We conjecture that the latter statement is never true, implying that Z is deformation equivalent to $(\mathrm{K3})^{[2]}$.

Relevant question: for X HK, H ample on X, is $\mathcal{O}_X(2H)$ globally generated?

Assuming this conjecture, we obtain an equivalence between local Torelli for $(\mathrm{K3})^{[2]}$ and global Torelli. So we study $LG//PGL(6) = \mathcal{M}$ equipped with a rational map p to $\mathbb{D}^{BB}$. Infinitesimal Torelli says that p is dominant, and deg $p < \infty$.