Positivity Properties of Divisors and Higher Codimension Cycles

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• Consider compact manifold M, and

$$\beta \in H_k(M, \mathbf{Z}).$$

Then $\exists$ "cycle" N, and $f : N \longrightarrow M$, such that

$$\beta = f_*[N].$$

Also,

$$-\beta = f_*[N^{\text{opposite orientation}}].$$

• In algebraic geometry, situation is very different. There are notions of positivity: e. g. can't simultaneously realize homology class and its opposite by algebraic subvarieties.

• Cones of positive cycles carry interesting information about underlying variety.

• Codimension one situation now quite well understood.

• Analogous questions in higher codimension just starting to attract attention.

Positivity in Codimension One

Set-up:

$$X \;=\; \text{smooth projective variety } /\mathbf{C}$$
$$\dim_{\mathbf{C}} X \;=\; n.$$

Definition: A divisor (or $(n-1)$-cycle) on X is

$$D \;=\; \sum a_i D_i$$

where:

$$D_i \;\subseteq\; X \quad \text{is irreducible, codim} = 1$$
$$a_i \;\in\; \mathbf{Z}.$$

(Can also consider $\mathbf{Q}$- or $\mathbf{R}$-divisors.)

D is effective if all $a_i \geq 0$.

Example 1. L a line bundle on X,

$$s \;\in\; \Gamma(X, L)$$

a section. Then take

$$D \;=\; \text{div}(s) \;=_{\text{def}} \Big\{ s = 0 \Big\}.$$

Moreover: D determines L and s (up to scalars). We write

$$L \ = \ \mathcal{O}_X(D).$$

Example 2. Consider

$$X \ \subseteq \ \mathbf{P}^r \quad , \quad H \ \subseteq \ \mathbf{P}^r \text{ a hyperplane.}$$

The divisor

$$D_H \ = \ X \cap H$$

is called a "hyperplane section" of X.

Can essentially recover $X \subseteq \mathbf{P}^r$ from $D = D_H$:

$$\exists \ s_0, \ldots, s_r \in \Gamma(X \, , \mathcal{O}_X(D)) \ \text{ s.t.}$$
$$X \hookrightarrow \mathbf{P}^r \ \text{ is } \ x \mapsto [s_0(X), \ldots, s_r(x)].$$

Such divisors give rise to the strongest notion of positivity:

Definition. D (or L) is very ample if it is a hyperplane section under a projective embedding.

It is hard to recognize very ample divisors. So we introduce another

Definition. L or D is ample if $\exists\ m \gg 0$ s.t. $L^{\otimes m}$ or mD is very ample.

Theorem. Let $L = \mathcal{O}_X(D)$. The TFAE:

(1). L is ample.

(2). [Nakai-Moishezon] $\forall$ irreducible $V \subset X$,

$$\left(D^{\dim V} \cdot V\right) \ =_{\text{def}} \int_V c_1(L)^{\dim V} \ > \ 0.$$

(3). [Kodaira] L has a hermitian metric h whose curvature $\Theta(L,h)$ is a positive $(1,1)$-form.

(4). [Serre] $\forall$ coherent sheaves $\mathcal{F}$ on X, have

$$H^i(X\,,\mathcal{F} \otimes \mathcal{O}_X(mD)) \ = \ 0$$

for $i > 0$, $m \geq m_0(\mathcal{F})$.

Note: Nakai-Moishezon implies amplitude of D depends only on

$$[D] \in H^2(X, \mathbf{R}).$$

But the conditions are non-linear in $[D]$.

Question: Is it enough to assume in Nakai-Moishezon that

$$(D \cdot C) > 0$$

for every irreducible curve C?

No, but:

Theorem [Kleiman]: If

$$\Big(D \cdot C\Big) \geq 0 \qquad (*)$$

for every curve C, then D is a limit of ample ($\mathbf{Q}$-) divisors.

Definition: D (or L) is nef if (*) holds.

Cones. Define

$$N^1(X)_{\mathbf{R}} \subseteq H^2(X, \mathbf{R})$$

to be subspace spanned by cohomology classes of all divisors. Inside $N^1(X)_{\mathbf{R}}$ one has two cones:

$$\text{Amp}(X) = \Big(\text{ cone spanned by ample divisors } \Big)$$
$$\text{Nef}(X) = \Big\{ \xi \,|\, (\xi \cdot C) \geq 0 \ \forall \ C \Big\}.$$

Kleiman's Theorem $\Longleftrightarrow$

$$\text{Nef}(X) = \overline{\text{Amp(X)}}$$
$$\text{Amp}(X) = \text{interior}\Big(\text{Nef}(X)\Big)$$

Dual Viewpoint: Look at

$$N_1(X)_{\mathbf{R}} = \Big\{ \text{ cohom. classes of 1-cycles } \Big\}$$
$$= \Big(N^1(X)_{\mathbf{R}}\Big)^{\vee}$$

$\text{NE}(X) =_{\text{def}}$ Cone spanned by effective curves

Then

$$\text{Nef}(X) = \overline{\text{NE}}(X)^{\vee}$$

are dual cones.

Upshot: Ample divisors $\longleftrightarrow$ Effective curves

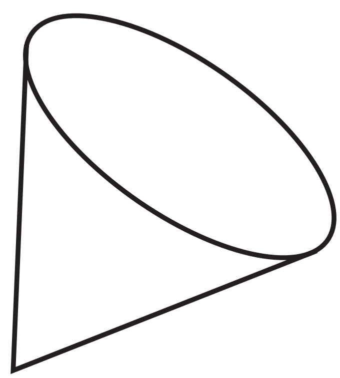
Example. Let E be an elliptic curve, take

$$X \;=\; E \times E.$$

Then

$$\mathsf{Nef}(X) \;=\; \overline{\mathsf{NE}}(X) = \;\{\xi \,|\, (\xi \cdot \xi) \geq 0\}$$

is a circular cone:



Example. If

$$X \;=\; C \times C \quad , \quad g(C) \geq 2$$

or

$X \;=\;$ Blow-up of $\mathbf{P}^2$ at > 9 very general points,

then Nef(X) and $\overline{\mathsf{NE}}(X)$ are unknown!

Conjecture: Let X be the blow-up of any surface at many very general points. Then nef and effective cones always have “circular part.”

Question: What is the structure of Nef(X) in general?

Theorem [Campana-Peternell]. In a neighborhood of almost all points in its boundary, Nef(X) is locally defined by a polynomial inequality of form

$$\left\{ \ (\xi^{\dim V} \cdot V) \geq 0 \ \right\}$$

for some subvariety $V \subseteq X$.

I.e. Nef(X) is almost everywhere bounded by algebraic hypersurfaces.

Ask: What can we learn about X from Nef(X) or $\overline{\text{NE}}(X)$?

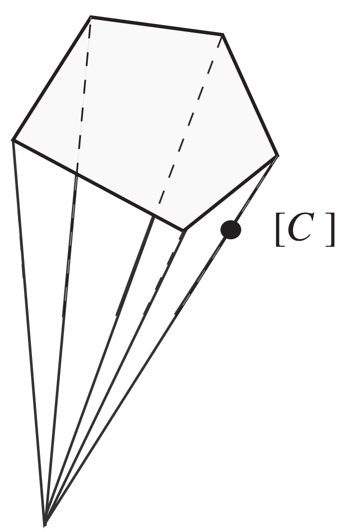
Example. Suppose that

$$\phi : X \longrightarrow Y$$

is a non-constant morphism, and $C \subseteq X$ is curve such that

$$\phi(C) \ = \ \text{point.}$$

Then $[C]$ lies on the boundary of $\overline{\text{NE}}(X)$:



Amazingly, converse true for "suitably-situated" curves.

Cone Theorem [Mori, ...] Let K_X be a canonical divisor of X, i.e.

$$K_X \;=\; \mathrm{div}(\eta) \quad , \quad \eta \in \Gamma(X, \Omega^n_X).$$

Suppose that C is curve whose class lies on boundary* of $\overline{\mathrm{NE}}(X)$, and assume that

$$\Big(C \cdot K_X\Big) \;<\; 0.$$

Then there is $\phi : X \longrightarrow Y$ contracting C.

• This is the beginning of the minimal model program.

• Illustrates principle that the deepest positivity information comes by taking into account the canonical bundle.

*Precise condition is that $[C]$ generates extremal ray of $\overline{\mathrm{NE}}(X)$.

Effective Divisors

Definition. The "pseudoeffective cone" of X

$$\overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbf{R}}$$

is closure of cone generated by all effective divisors.

Ex. When $\dim X = 2$, $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{NE}}(X)$.

NB: $\mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}}(X)$.

First Question: What is the interior of $\overline{\mathrm{Eff}}(X)$?

Definition. D is big if

$$\dim H^0\Big(X, \mathcal{O}_X(mD)\Big) \sim m^n.$$

Example: If A is ample and E is effective, then $D = A + E$ is big.

Classes of big divisors span the big cone

$$\mathrm{Big}(X) \subseteq N^1(X)_{\mathbf{R}}.$$

Theorem [Kodaira?]

$$\text{interior}\Big(\overline{\text{Eff}}(X)\Big) \ = \ \text{Big}(X).$$

Heuristic Principle: Interior of pseudoeffective cone consists of divisors that "move a lot."

Problem: Can one say anything about the boundary of $\overline{\text{Eff}}(X)$?

Variational Riemann–Roch and Volume Funct.

Riemann–Roch Problem: Study $h^0\Big(X, \mathcal{O}_X(mD)\Big)$ as a funciton of m.

Def.

$$\text{vol}_X(D) \ = \ \lim_{m\to\infty} \frac{h^0(mD)}{m^n/n!}.$$

Example. If D is ample (or nef), then

$$h^0(mD) \ = \ \frac{(D^n)}{n!} \cdot m^n + \text{LOT},$$

$$\therefore \quad \text{vol}_X(D) \ = \ (D^n).$$

Example: $\text{vol}_X(D) > 0 \iff D$ is big.

Example: $\exists$ big (integer) divisors D such that:

- $\text{vol}_X(D)$ arbitrarily small;
- [Cutkosky] $\text{vol}_X(D)$ irrational.

Note: $\text{vol}_X(D)$ makes sense for $\mathbf{Q}$-divisors.

Theorem [Laz., Boucksom]. $\text{vol}_X(D)$ depends only on the cohomology class of D, and it is computed by a (uniquely defined) continuous function

$$\text{vol}_X : N^1(X)_{\mathbf{R}} \longrightarrow \mathbf{R}.$$

This function is log-concave on $\text{Big}(X)$, i.e.

$$\text{vol}_X(\xi + \xi')^{1/n} \;\geq\; \text{vol}_X(\xi)^{1/n} + \text{vol}_X(\xi')^{1/n}$$

for all big ξ, ξ'.

Ask: What kind of function can vol_X be?

Example: In simple situations (eg. toric varieties) vol_X is piecewise polynomial. But it is not so in general.

Thm. [Laz.-Mustaţă, Boucksom-Favre-Jonsson] vol_X is $\mathcal{C}^1$ on $\text{Big}(X)$.

Problem: Is vol_X almost everywhere piecewise analytic?

What can one do with vol_X?

Recall: $\overline{\text{NE}}(X)$ is cone dual to $\text{Nef}(X)$.

Ask: What is cone in $N_1(X)_{\mathbf{R}}$ dual to $\overline{\text{Eff}}(X)$?

Define

$$\text{Mov}(X) \ \subseteq \ N_1(X)_{\mathbf{R}}$$

to be cone generated by classes of irreducible curves that move in a family generically covering X.

A moveable class evidently has non-negative intersection with any effective divisor.

Theorem [Boucksom-Demailly-Paun-Peternell]

$$\text{Mov}(X) = \overline{\text{Eff}}(X)^{\vee}.$$

Proof draws on Fujita Approximation Theorem, that one can approximate the volume of a divisor by the self-intersection of ample divisor on a modification.

Corollary [BDPP]. X is uniruled if and only if K_X is not pseudoeffective.

There is another remarkable result involving the canonical bundle.

Theorem [Tsuji, Hacon-McKernan, Takayama]. There exists a constant $C_n > 0$ depending only on n such that if X is any n-fold of general type, then

$$\text{vol}_X(K_X) \geq C_n.$$

Vaguely speaking, the Theorem says that the class of K_X is uniformly bounded away from the boundary of $\overline{\text{Eff}}(X)$.

As a corollary, the same authors establish a surprising uniformity property for the rational mappings defined by pluricanonical linear series.

Corollary. [T, H-McK, T] There is a constant B_n such that if X is an n-fold of general type, then the rational mapping

$$\varphi_{|mK|} : X \dashrightarrow \mathbf{P}$$

defined by $|mK_X|$ is birational onto its image for $m \geq B_n$.

Cycles of Higher Codimension

Notation: Write

$$N_p(X)_{\mathbf{R}} \ \subseteq \ H_{2p}(X, \mathbf{R})$$

for the subspace spanned by the classes of algebraic cycles, and

$$\overline{\mathsf{Eff}}_p(X) \ \subseteq \ N_p(X)_{\mathbf{R}}$$

for the closure of the cone determined by effective cycles.

Then set

$$\mathrm{Big}_p(X) \;=\; \mathrm{interior}\Big(\overline{\mathrm{Eff}}_p(X)\Big).$$

Ask: When is a p-cycle big?

Intuition: Cycle should be "positively embedded," or should "move a lot."

Peternell's Question: Suppose $Y \subseteq X$ is p-dimensional subvariety with ample normal bundle. Is then

$$[Y] \;\in\; \mathrm{Big}_p(X)\ ?$$

Voisin: Gave counter-examples for surfaces in a four-fold. In her example, Y moves in a family covering X.

Conjecture (Voisin): If Y moves in a family covering X, and with tangent spaces dominating the Grassmannian of p-planes in T_xX for general $x \in X$, then $[Y]$ is big.

Proposal: Should look for asymptotic characterization.

Conjecture (Ein-Laz.) Let $\beta \in N_p(X)_{\mathbf{Z}}$. Then ξ is big if and only if following holds:

> $\exists$ constant $C > 0$ and arbitrarily large integers k such that $k \cdot \beta$ is represented by an effective cycle Z_k whose support passes through
>
> $$\geq \ C \cdot k^{\frac{n}{n-p}}$$
>
> general points of X.

Remarks. • If β is big, then condition holds.

• Statement OK if $p = 1, n-1$

"Theorem" [Ein-Laz.] If Pic$(X) = \mathbf{Z}$, then Conjecture OK for $p = n-2$.

Question: Is there a naturally defined function

$$\mathrm{vol}_p : N_p(X)_{\mathbf{R}} \longrightarrow \mathbf{R}$$

that cuts out $\overline{\mathrm{Eff}}_p(X)$?

Grothendieck's Questions

Definition (Grothendieck) Define

$$\mathrm{Nef}^p(X) \ \subseteq \ N^p(X)_{\mathbf{R}}$$

to be the set of codimension p cycles that have non-negative intersection number with all pseudo-effective p-cycles. I.e.

$$\mathrm{Nef}^p(X) \ = \ \Big(\overline{\mathrm{Eff}}_p(X)\Big)^{\vee}.$$

Grothendieck raised the

Question. Is the intersection of nef classes of codimensions p and q a nef class of codimension $p+q$?

Would follow if one knew

$$\mathsf{Nef}^p(X) \ \subseteq \ \overline{\mathsf{Eff}}^p(X),$$

but this isn't known (or maybe even expected).

Example. Let E be a nef vector bundle. Then the Chern classes (and Schur polynomials in the Chern classes) of E are nef. But not known whether

$$c_p(E) \ \in \ \overline{\mathsf{Eff}}^p(X).$$