Rational curves on K3 surfaces

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§1 The Mori-Mukai argument

A polarized K3 surface (S, f) consists of a K3 surface and an ample divisor f that is primitive in the Picard group. Its degree is the positive even integer $f \cdot f$.

$\mathcal{K}_g, g \geq 2$ denotes the moduli space (stack) of complex polarized K3 surfaces of degree $2g-2$, which is smooth and connected of dimension 19.

$g = 2:$ $\quad S = \{w^2 = F_6(x, y, z)\}$, a double cover of $\mathbb{P}^2$ branched over a sextic; f is the pull-back of the polarization on $\mathbb{P}^2$

$g = 3:$ $\quad S = \{F_4(w, x, y, z) = 0\}$, a quartic surface in $\mathbb{P}^3$

$g = 4:$ $\quad S = \{F_2 = F_3 = 0\}$, a complete intersection of a quadric and a cubic in $\mathbb{P}^4$

The following is attributed to Mumford, although it was known to Bogomolov around the same time:

Theorem 1 (Mori-Mukai 1983) *Every complex projective K3 surface contains at least one rational curve. Furthermore, suppose* $(S, f) \in \mathcal{K}_g$ *is very general, i.e., in the complement of a countable union of Zariski-closed proper subsets. Then* S *contains an infinite number of rational curves.*

Idea: Let N be a positive integer. Exhibit a K3 surface $(S_0, f) \in \mathcal{K}_g$ and smooth rational curves $C_i \to S_0$, with $[C_1 \cup C_2] = Nf$ and $[C_i] \not\sim f$.

Deform $C_1 \cup C_2$ to an irreducible rational curve in nearby fibers.

Kummer construction (for $N = 1$)
We exhibit a K3 surface S_0 containing two smooth rational curves C_1 and C_2 meeting transversally at $g + 1$ points.

E_1 and E_2 elliptic curves admitting an isogeny $E_1 \to E_2$ of degree $2g + 3$ with graph $\Gamma \subset E_1 \times E_2$; $p \in E_2$ a 2-torsion point

Take associated Kummer surface

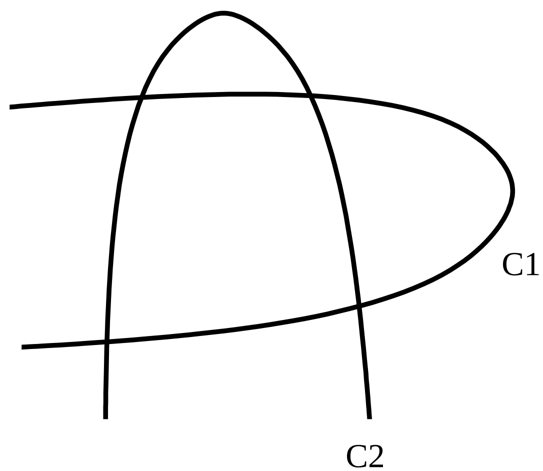
$$S_0 = (E_1 \times E_2)/\langle \pm 1 \rangle$$

Γ intersects $E_1 \times p$ transversally in $2g + 3$ points, one of which is 2-torsion in $E_1 \times E_2$

Take C_1 and C_2 to be the images of Γ and $E_1 \times p$ in S_0, smooth rational curves meeting transversally in $g + 1$ points.

Sublattice of Pic(S_0) determined by C_1 and C_2:

	C_1	C_2
C_1	-2	$g+1$
C_2	$g+1$	-2



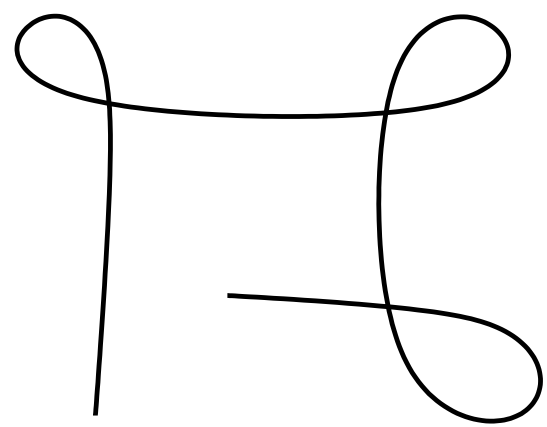
Two smooth rational curves in S_0

$f = C_1 + C_2$ is big and nef and has no higher cohomology (by Kawamata-Viehweg vanishing)

Deform (S_0, f) to a polarized $(S, f) \in \mathcal{K}_g$

$$\mathcal{S} \to B, \quad \mathrm{dim}(B) = 1$$

with f ample and indecomposable in the effective monoid



Deformation of $C_1 \cup C_2$ in nearby K3 surface

$H^i(\mathcal{O}_{S_0}(f)) = 0, i > 0$ thus $C_1 \cup C_2$ is a specialization of curves in the generic fiber and Def$(C_1 \cup C_2 \subset \mathcal{S})$ is smooth of dimension $g + 1$.

The locus in Def$(C_1 \cup C_2 \subset \mathcal{S})$ parametrizing curves with at least ν nodes has dimension $\geq g+1-\nu$. When $\nu = g$ the corresponding curves are necessarily rational.

The fibers of $\mathcal{S} \to B$ are not uniruled and thus contain a finite number of these curves, so the rational curves deform into nearby fibers

Conclusion For $(S, f) \in \mathcal{K}_g$ generic, there exist rational curves in the linear series $|f|$

However, rational curves can only specialize to unions of rational curves (with multiplicities), thus *every* K3 surface in $\mathcal{K}_g$ contains at least one rational curve

Remark 2 Yau-Zaslow, Beauville, Bryan-Leung, Xi Chen, etc. have beautiful *enumerative results* on the rational curves in $|f|$

Generalized construction (for arbitrary N)

(S_0, f) polarized K3 surface of degree $2g-2$ with

$$\mathrm{Pic}(S_0)_{\mathbb{Q}} = \mathbb{Q}C_1 + \mathbb{Q}C_2$$

with C_1 and C_2 smooth rational curves and

$$Nf = C_1 + C_2$$

	C_1	C_2
C_1	-2	$N^2(g-1)+2$
C_2	$N^2(g-1)+2$	-2

The existence of these can be deduced from surjectivity of Torelli, i.e., take a general lattice polarized K3 as above

Deform (S_0, f) to a polarized $(S, f) \in \mathcal{K}_g$ as above

$$\mathcal{S} \to B, \quad \dim(B) = 1.$$

Def$(C_1 \cup C_2 \subset \mathcal{S})$ is smooth of dimension $N^2(g-1)+2$; the locus parametrizing curves with at least $N^2(g-1)+1$ nodes (i.e., the rational curves) has dimension ≥ 1.

There are a finite number in each fiber, thus we obtain *irreducible* rational curves in $|Nf|$ for generic K3 surfaces in $\mathcal{K}_g$

This argument proves that *very general* K3 surfaces admit irreducible rational curves in $|Nf|$ for each $N \in \mathbb{N}$

In particular, they have admit infinitely many rational curves

Conceivably, for special K3 surfaces these might coincide, i.e., so that the infinite number of curves *all* specialize to cycles

$$m_1 C_1 + \ldots + m_r C_r$$

supported in a *finite* collection of curves

Remark 3 N.C. Leung, J.H. Lee, B.S. Wu, and Jun Li have enumerated curves in $|2f|$, analyzing the contributions of reducible and non-reduced rational curves

A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger have recently shown that the BPS count of rational curves in $|Nf|$ (i.e., the number from Gromov-Witten theory taking the multiple cover formula into account) depends only on the self intersection

$$(Nf) \cdot (Nf) = N^2(2g-2),$$

not on the divisibility N

§2 Questions on rational curves

K algebraically closed field of characteristic zero

T/K projective K3 surface

The following is well-known but hard to trace in the literature:

Question 4 (Main conjecture) There exist an infinite number of rational curves on T.

The following extreme version is more easily attributed:

Conjecture 5 (Bogomolov 1981) Let S be a K3 surface defined over number field F. Then each point $s \in S(F)$ is contained in a rational curve $C \subset S$ defined over $\overline{\mathbb{Q}}$.

It would follow that $S_{\overline{\mathbb{Q}}}$ has an infinite number of rational curves, because $S(\overline{\mathbb{Q}})$ is Zariski dense in S.

Moreover, we can easily reduce the Main Conjecture to the case of number fields.

Proposition 6 (Bloch, Voisin, Ran, etc.) *Let B be a smooth complex variety, $\pi : \mathcal{T} \to B$ a family of K3 surfaces, and $\mathcal{D}$ a divisor on $\mathcal{T}$. Then the set*

$$V := \{b \in B : \text{there exists a rational curve } C \subset \mathcal{T}_b = \pi^{-1}(b) \text{ with } [C] = \mathcal{D}_b\}.$$

is open. More precisely, any generic immersion

$$f_b : \mathbb{P}^1 \to \mathcal{T}_b, \quad f_{b*}[\mathbb{P}^1] = \mathcal{D}_b,$$

can be deformed to nearby fibers.

Thus rational curves deform provided their homology classes remain of type $(1,1)$. (**Note use of Hodge theory!**)

Proof: Main Conjecture/$\overline{\mathbb{Q}}$ $\Rightarrow$ Main Conjecture/K

Suppose there exists a K3 surface T over K with a finite number of rational curves. We may assume that K is the function field of some variety $B/\overline{\mathbb{Q}}$. 'Spread out' to get some family $\mathcal{T} \to B$, and choose a point $b \in B(\overline{\mathbb{Q}})$ such that the fiber $\mathcal{T}_b$ has general Picard group

$$\mathrm{Pic}(\mathcal{T}_b) = \mathrm{Pic}(T_{\overline{K}}).$$

Since $\mathcal{T}_b$ has a infinite number of rational curves, the same holds for T.

§3 Rational curves on special K3 surfaces

Theorem 7 (Bogomolov-Tschinkel 2000) *Let S be a complex projective K3 surface admitting either*

1. *a non-isotrivial elliptic fibration; or*

2. *an infinite group of automorphisms.*

Then S admits an infinite number of rational curves.

The argument actually goes through for all but the most degenerate elliptic K3 surfaces, which turn out to be either Kummer elliptic surfaces or to have Néron-Severi group of rank twenty.

Proof: in the case $|\mathrm{Aut}(S)| = \infty$

Consider the monoid of effective divisors on S

Each nonzero indecomposable element D contains rational curves by the Mori-Mukai argument (when $D \cdot D > 0$) or direct analysis (when $D \cdot D = 0, -2$)

It suffices to show there must be infinitely many such elements

This is clear, because otherwise the image of

$$\mathrm{Aut}(S) \to \mathrm{Aut}(\mathrm{Pic}(S))$$

would be finite, so the kernel would have to be infinite, which is impossible

Example:

Let Λ be a rank-two even lattice of signature $(1,1)$ that does not represent -2 or 0

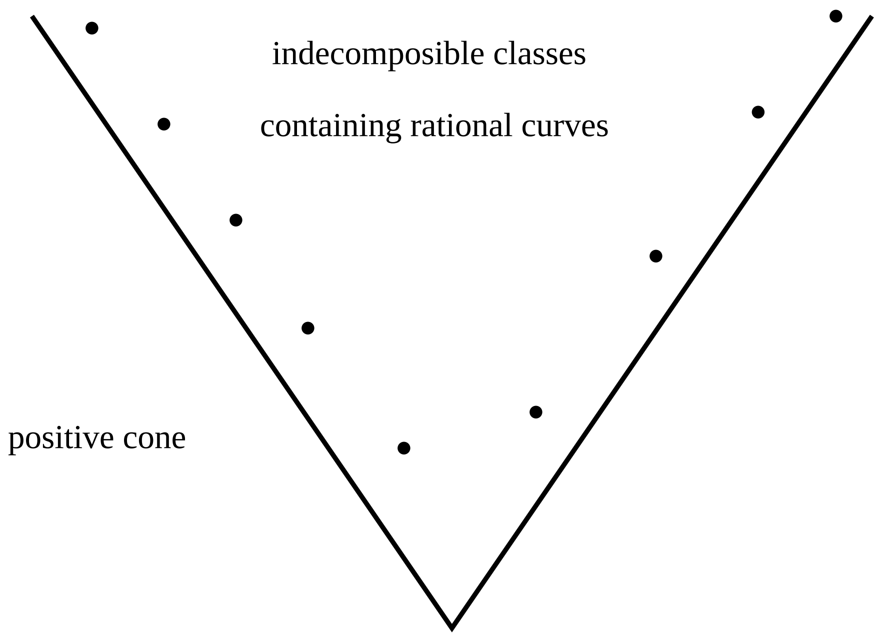
(S, f) polarized K3 surface with $\mathrm{Pic}(S) = \Lambda$

The positive cone

$$\mathcal{C}_S := \{D \in \Lambda : D \cdot D > 0, D \cdot f > 0\}$$

equals the ample cone and is bounded by irrational lines

Infinitely-many indecomposable effective divisors $\Rightarrow$ infinitely-many rational curves in S



'Typical' rank-two K3 surfaces have infinitely many curve classes containing rational curves

Note: K3 surfaces with Aut(S) infinite or admitting an elliptic fibration have

$$\text{rank}(\text{Pic}(S)) \geq 2$$

Thus these techniques do not apply to 'most' K3 surfaces

Indeed, I know no example in the literature of a K3 surface $S/\overline{\mathbb{Q}}$ with Pic(S) $= \mathbb{Z}$ admitting infinitely many rational curves

This is entirely consistent with the possibility that the Mori-Mukai argument might break down over a countable union of subvarieties in $\mathcal{K}_g$

§4 Mori-Mukai in mixed characteristic

joint work with F. Bogomolov and Y. Tschinkel

S projective K3 surface over a number field F with $\overline{S} = S_{\overline{\mathbb{Q}}}$

$\mathfrak{o}_F$ ring of integers with spectrum $B = \mathrm{Spec}(\mathfrak{o}_F)$

$\pi : \mathcal{S} \to B$ a flat projective model for S

$\mathfrak{p} \in B$ a prime of good reduction for S, i.e., $\mathcal{S}_{\mathfrak{p}} = \pi^{-1}(\mathfrak{p})$ is a smooth K3 surface over a *finite* field

k finite field with algebraic closure $\overline{k}$, $p = \mathrm{char}(k)$

X/k K3 surface and $\overline{X} = X_{\overline{k}}$

Fr Frobenius endomorphism on $\bar{X}$ acting on ℓ-adic cohomology

$$\mathrm{Fr}^* : H^2(\overline{X}, \mathbb{Q}_\ell) \to H^2(\overline{X}, \mathbb{Q}_\ell)$$

X is *ordinary* if $p \nmid \mathrm{Trace}(\mathrm{Fr})$

Joshi-Rajan and Bogomolov-Zarhin have shown

$$\{\mathfrak{p} \in B : \mathcal{S}_\mathfrak{p} \text{ ordinary } \}$$

has positive Dirichlet density; we call these places of *excellent reduction*

Key properties: Assume X/k is an ordinary K3 surface

1. X is *not* uniruled, i.e., rational curves in X cannot deform;

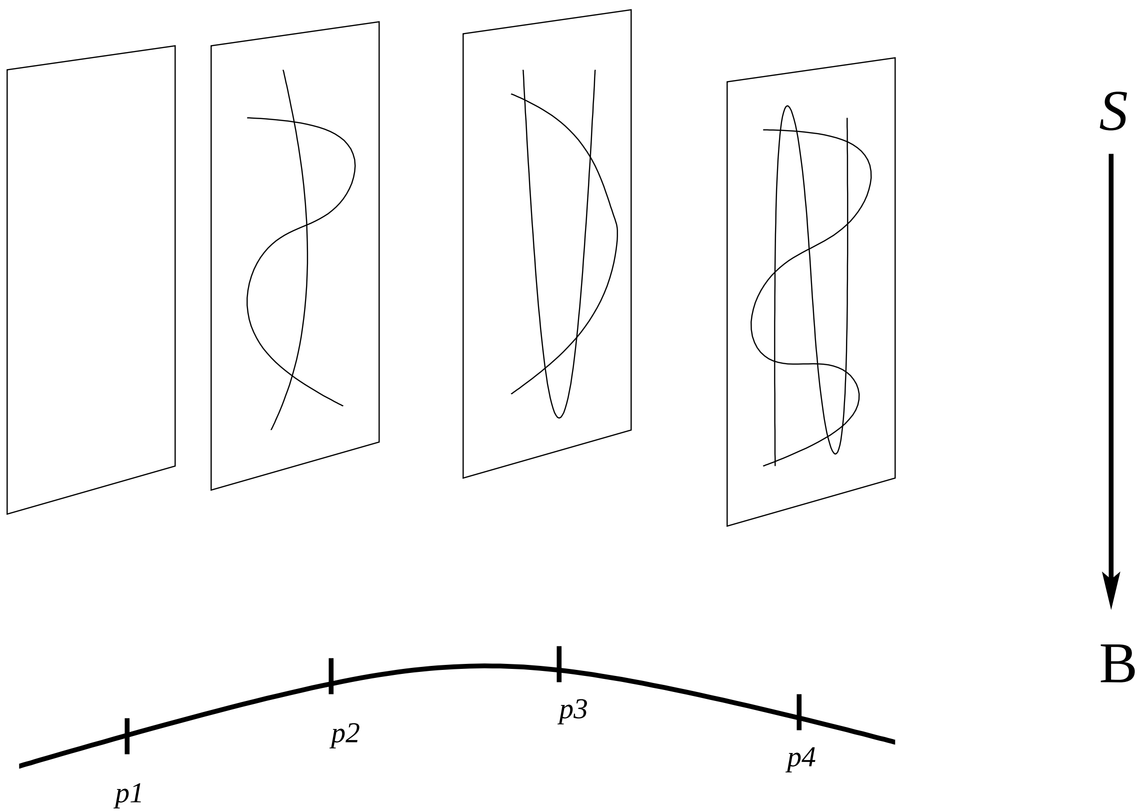
2. the Tate conjecture holds for X, thus the image of

$$\text{Pic}(\overline{X}) \to H^2(\overline{X}, \mathbb{Q}_\ell)$$

   consists of the elements fixed by Fr^m for some $m \in \mathbb{N}$;

3. $\text{Pic}(\overline{X})$ has rank ≥ 2.

Indeed, since Fr respects the intersection form, if α is a root of the characteristic polynomial of Fr then so is q^2/α



Reductions of a K3 surface mod $\mathfrak{p}$ have extra curve classes

S K3 surface over a number field with $\mathrm{Pic}(\overline{S}) = \mathbb{Z}f$

Goal: produce irreducible rational curves in $\overline{S}$ with class Nf for unbounded N

Strategy: Part I

Choose an excellent place $\mathfrak{p}$ such that we can write

$$Nf|\mathcal{S}_{\mathfrak{p}} \equiv m_1C_1 + \ldots + m_rC_r \tag{1}$$

where the $C_i \subset \mathcal{S}_{\mathfrak{p}}$ are rational curves and each effective sub-cycle

$$n_1C_1 + \ldots + n_rC_r \prec m_1C_1 + \ldots + m_rC_r$$

is not proportional to f

Any curve $C \subset \overline{S}$ specializing to $m_1C_1 + \ldots + m_rC_r$ would have to be irreducible

For example, take $N \in \mathbb{N}$ *minimal* such that $Nf|\mathcal{S}_\mathfrak{p}$ can be expressed as a positive sum of indecomposable curves $C_i \subset \mathcal{S}_\mathfrak{p}$ *not* proportional to f

We prove that $N \to \infty$ as $\mathfrak{p} \to \infty$

Given $d > 0$,

$$\{\mathfrak{p} \in B : \mathcal{S}_{\mathfrak{p}} \supset C \text{ with } C \cdot f \leq d, C \not\sim f\}$$

is *finite*

Indeed, there are a finite-number of Noether-Lefschetz divisors corresponding to K3 surfaces T with

$$\mathrm{Pic}(T) \supset \Lambda, \qquad \Lambda = \begin{array}{c|cc} & f & C \\ \hline f & 2g-2 & d \\ C & d & C \cdot C \end{array},$$

and $B = \mathrm{Spec}(\mathfrak{o}_F)$ meets these in a finite number of primes

Consequently, as $\mathfrak{p} \to \infty$ both $d \to \infty$ and $N \to \infty$

Strategy, Part II: Emulate Mori-Mukai

Try to show that the cycle $m_1C_1 + \ldots + m_rC_r \subset \mathcal{S}$ deforms in a one-parameter family of *rational curves* in $\mathcal{S}$

This is relatively straightforward when

$$m_1 = \ldots = m_r = 1,$$

since we can apply the deformation theory arguments above with minor technical modifications

Note that the formal deformation space of $X = \mathcal{S}_\mathfrak{p}$ is smooth of dimension twenty over the Witt vectors; deformations of a polarized K3 surface are flat of dimension nineteen over the Witt vectors

Since $\mathcal{S}_\mathfrak{p}$ is ordinary it is not uniruled, so the one-parameter family of rational curves in $\mathcal{S}$ cannot all lie in $\mathcal{S}_\mathfrak{p}$

Thus this family dominates B, which tells us that the rational cycle lifts to a rational curve in characteristic zero

Standard algebraization arguments yield a rational curve in $\overline{S}$ with class Nf

A sample theorem:

Theorem 8 *Let S be a K3 surface defined over a number field with* $\mathrm{Pic}(\overline{S}) = \mathbb{Z}f$ *and* $f \cdot f = 2$. *Then S contains an infinite number of rational curves.*

In other words, S is a double cover of $\mathbb{P}^2$ branched over a very general plane sextic curve.

Here the associated involution greatly simplifies the multiplicity analysis

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11AM – 12PM.
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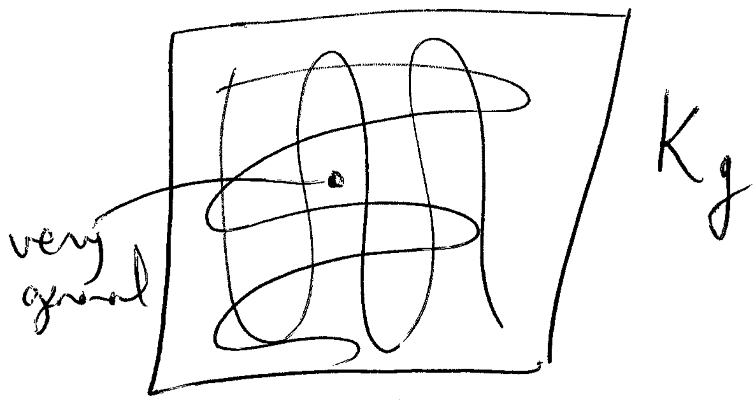
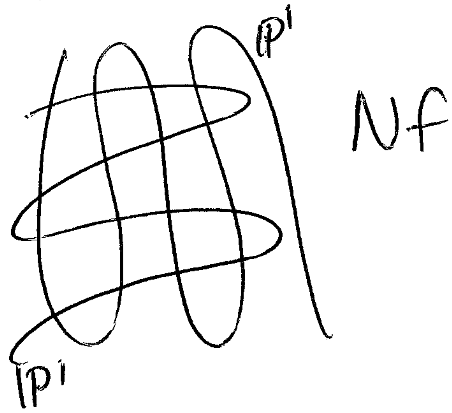
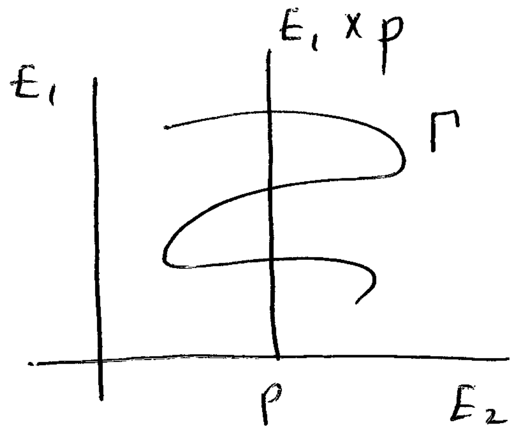
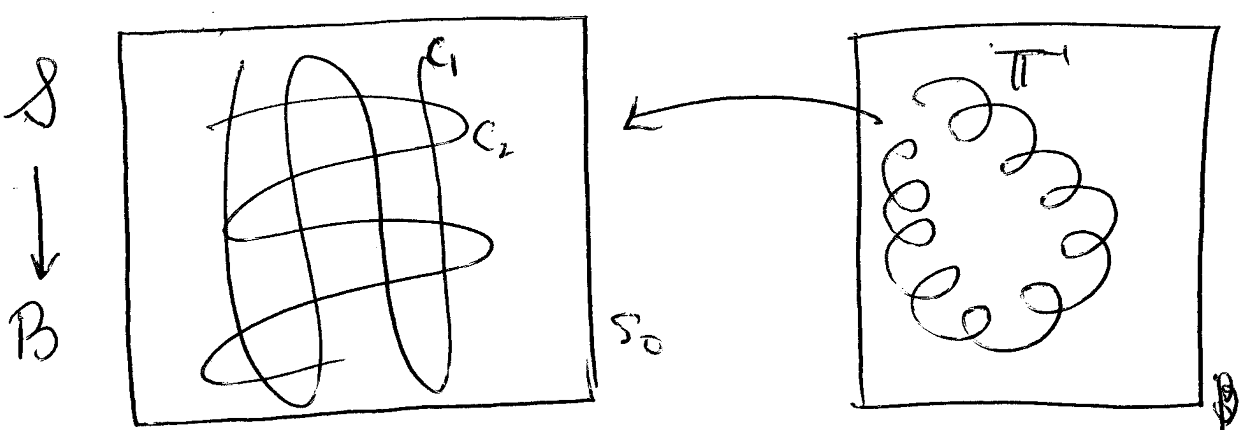
$f \cdot f = 2g - 2$









$(S, f) \in K_g$