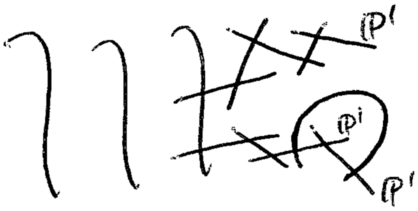
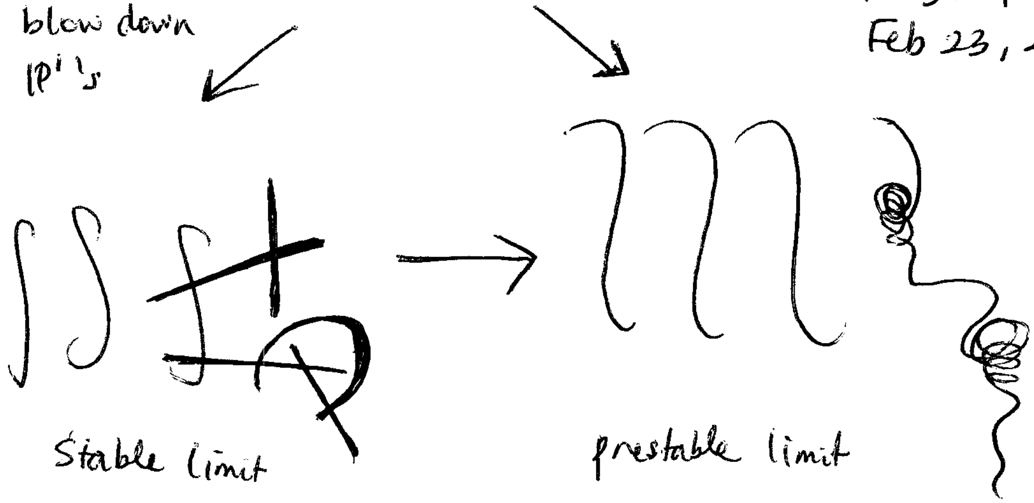
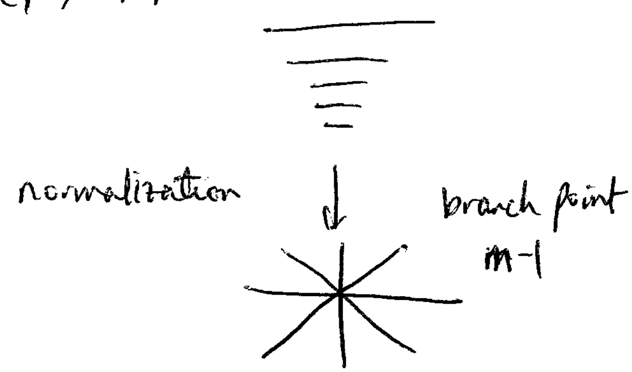
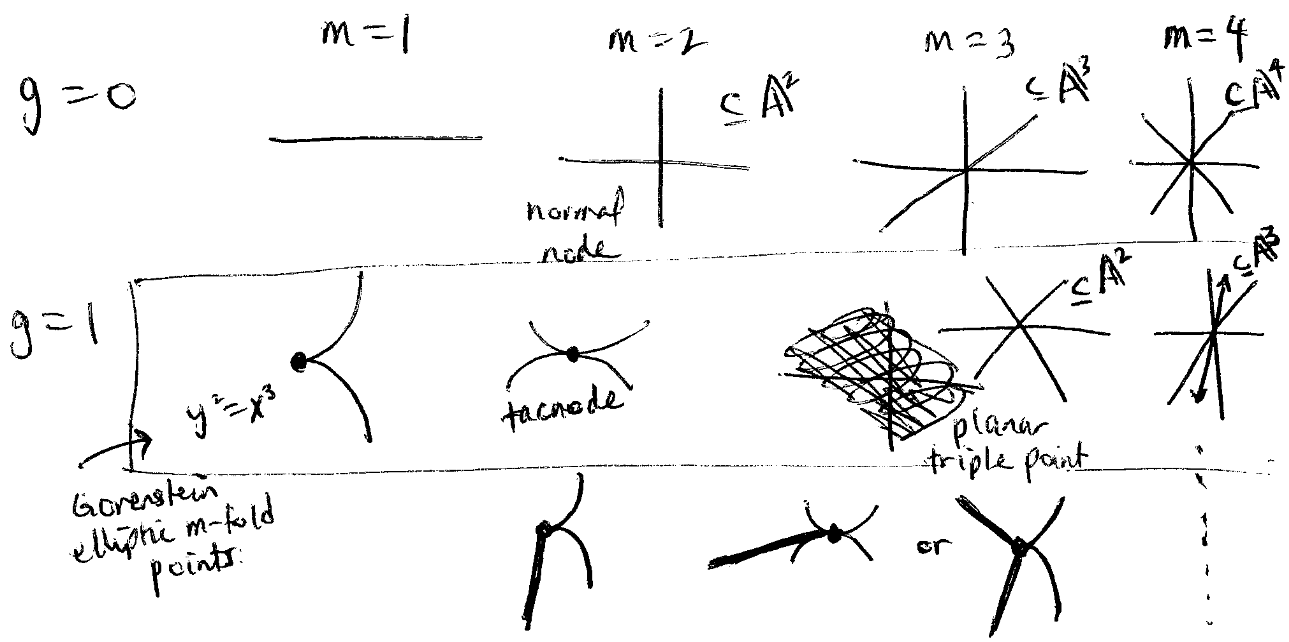
David Smyth
2-3PM
Feb 23, 2009

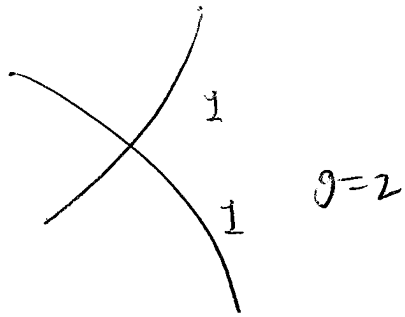




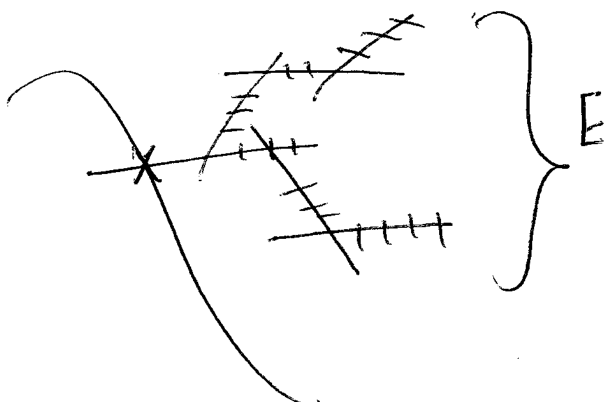
$P \in C$ singular point. Then we define
$g(P) = \delta(P) - m(P) + 1$



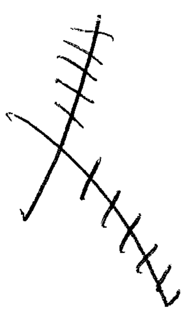


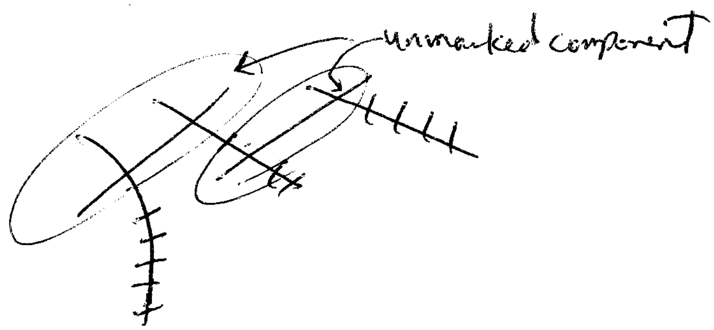


Fix $k \in \mathbb{Z}$.
$p_a = 0$ subcurve, $|E \cap E^c| = 1$, $\leq k$-marked points,



This is $\overline{M}_{g,A}$ with
$A = \{\frac{1}{k}, \dots, \frac{1}{k}\}$





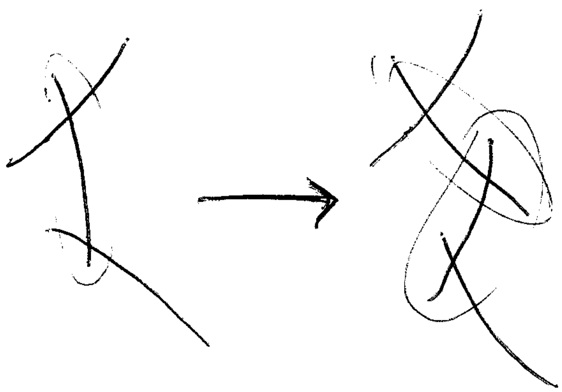
How do you make this into a classification? $\mathcal{C}_e^s \xrightarrow{\phi} \mathcal{C}$, $\mathcal{C}_e^s \to S$, $\mathcal{C} \to S$

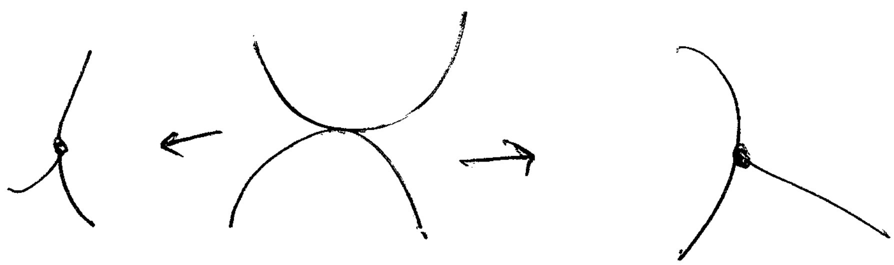
$$\begin{tikzcd}
\mathcal{C}_e^s \arrow[d] & \mathcal{C}^s \arrow[r, dashed] \arrow[dr] & \mathcal{C} \arrow[d] & \mathcal{C}_e \arrow[d] \\
& & T \arrow[dll] \arrow[dr] & \\
\overline{M}_{g,n} \subseteq & U & \subseteq & X
\end{tikzcd}$$

$$\begin{tikzcd}
s \in S \arrow[d, "\text{proper alteration}"] \\
T \arrow[d, dash] \\
\overline{M}_{g,n}
\end{tikzcd}$$

$$\mathrm{Exc}(\phi_S) \subseteq \mathcal{C}_S^S$$

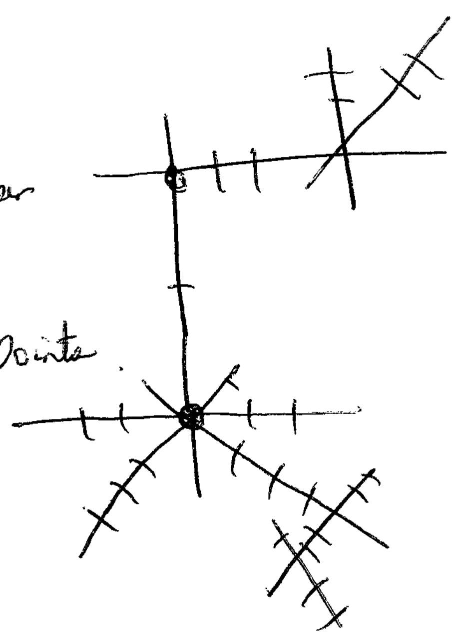
$$\Delta \longrightarrow \overline{M}_{g,n} \ni x \qquad \xrightarrow{\exists\, u_1, \dots, u_R \text{ lifts}} \quad \left.\begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_R \end{matrix}\right\} R \text{ connected components, fibers over } x.$$



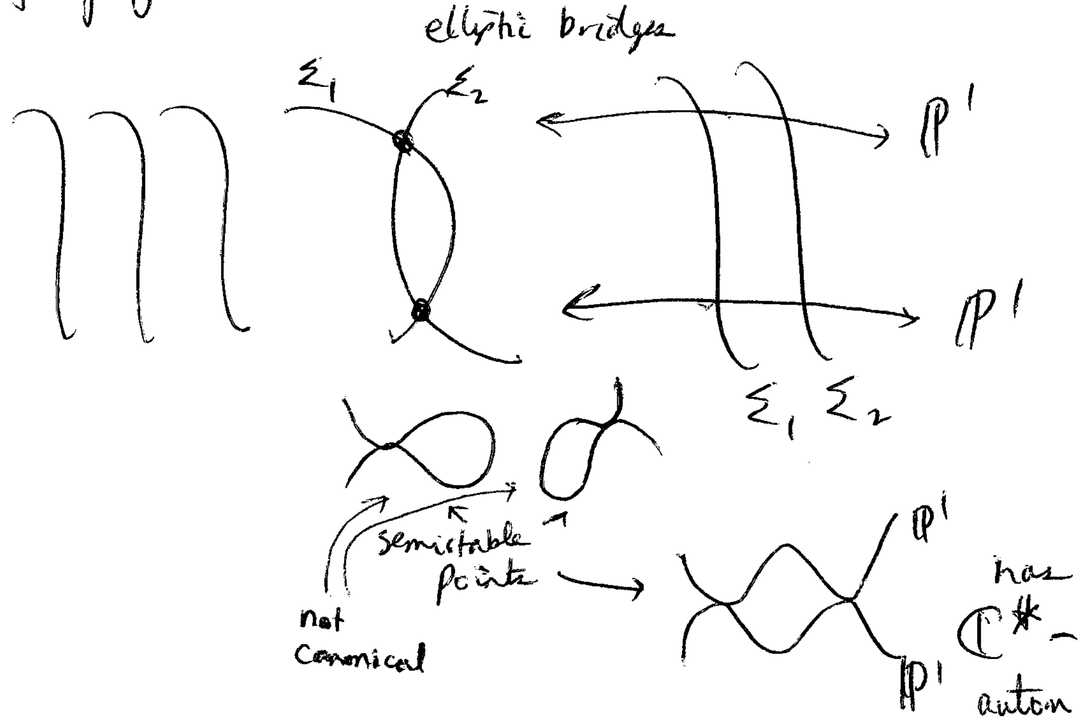


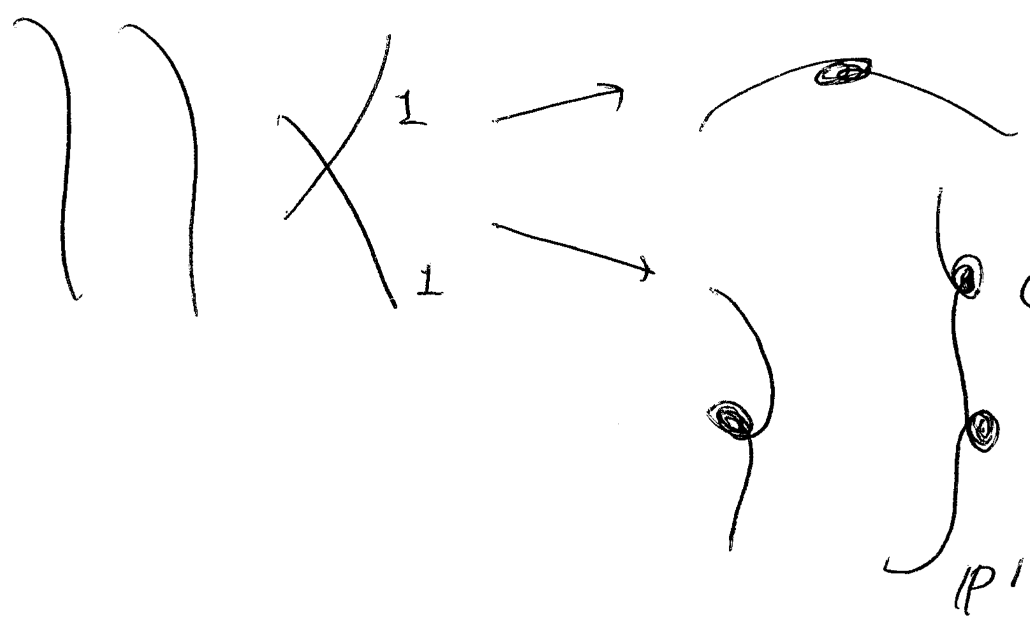
$0 \qquad \mathbb{C}^* \qquad \infty$

have finite/proper aut. gp $\iff$ $\exists$ only 2 marked points.



family of $g=3$ curves



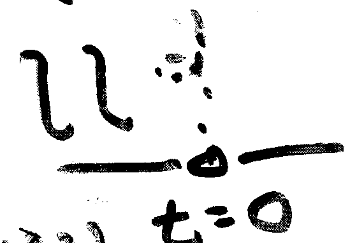


Towards a Classification of Modular Compactifications of $M_{g,n}$

MSRI Workshop: "Modern Moduli Theory"
David Smyth, Feb. 2009

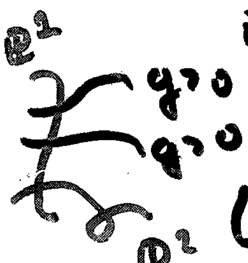
Q: What was wrong w/ classical moduli theory

A: M_g is not compact! i.e. $\exists$ families of smooth curves over punctured disc that do not extend to smooth family over disc. e.g. $y^2 = x^3 + x^2 + t$



Refn: a (connected, reduced) curve ($g \geq 2$) is stable if: a) C has only nodes ($xy=0$)

b) ω_C ample.



[Equiv. every rational component of $\tilde{C}$ has ≥ 3 distinguished points]

Class of stable curves has two essential properties:

1. Deformation-Open
2. Unique-Limit Property

1 + 2. $\Rightarrow$ Space of stable curves gives a compactification $\overline{M}_{g,n} \supset M_{g,n}$ (Deligne-Mumford '78)

Modular Compactifications of $\mathcal{M}_{g,n}$:

Let $\mathcal{U}_{g,n}$:= moduli stack of (all) n-pointed curves of $p_a = g$

i.e. objects are flat proper finitely-presented morphisms w/ n sections, whose geometric fibers are connected reduced 1-dim. schemes.

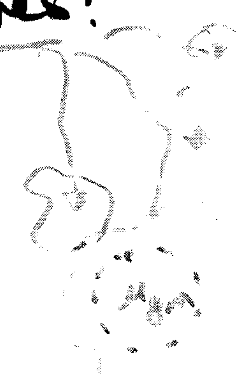
$$\begin{array}{c} \mathcal{C} \\ \downarrow \uparrow \{\sigma_i\}_{i=1}^{n} \\ T \end{array}$$

(Note: $\mathcal{C}$ is allowed to be an algebraic space.)

$\Rightarrow$ $\mathcal{U}_{g,n}$ is algebraic, locally of finite-type over $\mathbb{Z}$. (Jack Hall)

<u>Defn</u>: a <u>modular</u> <u>compactification</u> of $\mathcal{M}_{g,n}$ is an open substack $\mathcal{X} \subset \mathcal{U}_{g,n}$ s.t. $\mathcal{X}$ proper/$\mathbb{Z}$.

<u>Problem</u>: $\mathcal{U}_{g,n}$ is not irreducible when $g \geq 3$ (Zink...)
i.e. $\exists$ <u>non-smoothable curves</u>!



<u>Ad-hoc solution</u>: Let $\mathcal{V}_{g,n} \subset \mathcal{U}_{g,n}$ be component containing $\mathcal{M}_{g,n}$. Replace $\mathcal{U}_{g,n}$ by $\mathcal{V}_{g,n}$ in above defn.

<u>Defn</u>:
$(C, p_1, \dots, p_n)$ is <u>prestable</u> (<u>presemistable</u>) if every rational component of $\tilde{C}$ has ≥ 3 (≥ 2) distinguished points.

A modular compactification $\mathcal{X}$ is <u>stable</u> (<u>semistable</u>) if every geometric point $[C, p_1, \dots, p_n] \in \mathcal{X}$ is <u>prestable</u> (<u>presemistable</u>).

<u>Remark</u>:
$\exists$ 3 <u>stable</u> mod. compactifications in the literature.
$\overline{\mathcal{M}}_{g,n}$ (D-M), $\overline{\mathcal{M}}_g^{ps}$ (Schubert), $\overline{\mathcal{M}}_{g,A}$ (Hassett)

Two Goals:

1. Classify modular compactifications of $M_{g,n}$
2. Use them to study geometry of $\overline{M}_{g,n}$.

Remainder of this lecture...

1) Give an explicit classification of stable modular compactifications of $M_{g,n}$.

   $\Rightarrow$ upshot: There are tons of these, but they fail to give a completely satisfactory theory.

   (e.g. $\nexists$ stable modular compactification of M_g w\ only nodes ($xy=0$), cusps ($y^2=x^3$), tacnodes ($y^2=x^4$))

2) Investigate semistable compactifications:

   - $g=0$; every modular compactification is stable, i.e. we have a complete classification of mod. comp. of $M_{0,n}$.
   - $g=1$: we'll exhibit a sequence of semistable compactifications of $M_{1,n}$: $\overline{M}_{1,n}(1), \dots, \overline{M}_{1,n}(n-1)$ w\ beautiful properties: e.g. explicit description of boundary, explicit intersection theory...
   - $g \geq 2$: Existence of semistable compactifications is unknown, but I'll discuss work of Hassett-Hyeon, van der Wyck, Alper giving ideas how to proceed...

3) Connection to Birational Geometry & MMP:

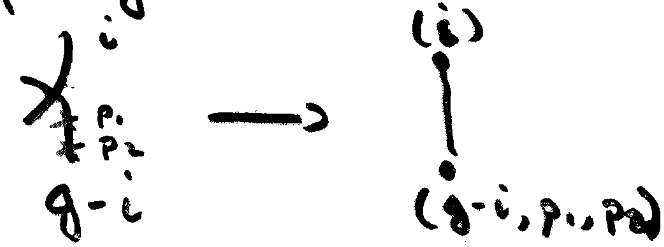
   Use modular compactifications to give Mori-chamber Decompositions of Effective Cones of $\overline{M}_{0,n}$, $\overline{M}_{1,n}$.

   In particular, get complete description of Log-MMP for $\overline{M}_{0,n}$ (Simpson, Fedorchuk-, Alexeev-Swinarski)

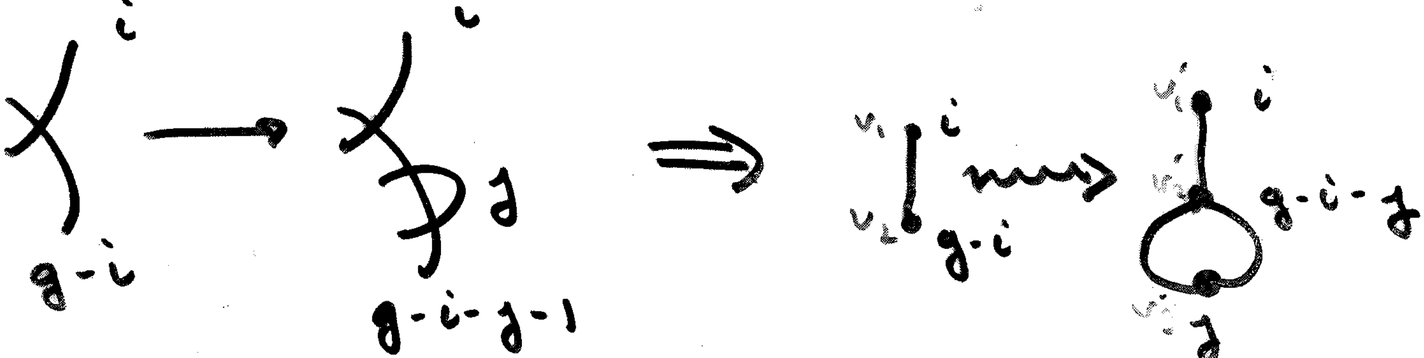
   $\overline{M}_{1,n}$ (In progress)

Combinatorial Data defining a stable mod. comp

Recall that any stable curve has a dual graph encoding its topological type:



We may define specialization of dual graphs in the obvious way, i.e. $G \rightsquigarrow G'$ if there's a corresponding 1-parameter specialization of stable curves.



$v_1 \rightsquigarrow v_1'$
$v_2 \rightsquigarrow v_2' \cup v_3'$

<u>Defn</u>: Let $G_1, \dots, G_N$ be an enumeration of dual graphs of n-pointed genus g stable curves. An <u>extremal assignment</u> (for $\overline{M}_{g,n}$) is an assignment

$$G_i \longmapsto \mathcal{Z}(G_i) \subset G_i \quad \forall i$$

satisfying:

a) $\mathcal{Z}(G_i) \neq G_i$

b) $\mathcal{Z}(G_i)$ is $\mathrm{Aut}(G_i)$-invariant

c) For any specialization $G \rightsquigarrow G'$ inducing $v \rightsquigarrow v_1' \cup \dots \cup v_k'$

$$v \in \mathcal{Z}(G) \Leftrightarrow v_1', \dots, v_k' \in \mathcal{Z}(G')$$

<u>Remark</u>: This is pure combinatorics, so one can (in principle) write <u>down</u> all extremal assignments for $\overline{M}_{g,n}$.

Defn: Given an extremal assignment $\mathcal{Z}$, a smoothable curve $(C, p_1, \dots, p_n)$ is $\mathcal{Z}$-stable if $\exists$ stable curve $(C^s, p_1^s, \dots, p_n^s)$ and a morphism $\phi: C^s \to C$ satisfying:

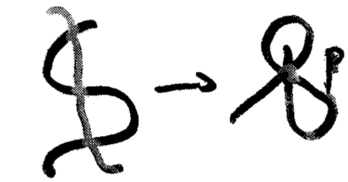
a) ϕ is surjective w/ connected fibers.

b) ϕ is an iso. on $C^s - \mathcal{Z}(C^s)$

c) If $Z_1, \dots, Z_k$ are connected components of $\mathcal{Z}(C^s)$, then $p_i := \phi(Z_i) \in C$ satisfies:

(# branches) $m(p_i) := |Z_i \cap Z_i^c|$

(genus) $g(p_i) := p_a(Z_i)$



Theorem: 1) Given extremal assignment $\mathcal{Z}$, $\overline{M}_{g,n}(\mathcal{Z}) := \{\mathcal{Z}\text{-stable curves}\} \subset \mathcal{V}_{g,n}$ is a stable modular compactification of $M_{g,n}$

2) If $\mathcal{X} \subset \mathcal{V}_{g,n}$ is a stable modular co s.t. then $\exists$ extremal assignment $\mathcal{Z}$ s.t. $\mathcal{X} = \overline{M}_{g,n}(\mathcal{Z})$.

Examples:

1) $\mathcal{Z}(C, \{p_i\}) = \{E \subset C \mid E \text{ unmarked}, p_a(E) = 1, |E \cap E^c| = 1\}$
$(g > 2 \text{ or } n > 1)$ i.e. "elliptic tails"

2) $\mathcal{Z}(C, \{p_i\}) = \{E \subset C \mid p_a(E) = 0, |E \cap E^c| = 1, E \text{ has} \leq k \text{ marked pts}\}$
$(g > 0 \text{ or } n > 2k)$ i.e. "$\leq k$-marked rational tails"

3) $\mathcal{Z}(C, \{p_i\}) = \{E \subset C \mid E \text{ unmarked}\}$
$(n > 2)$

Lemma: Let $\mathcal{L}$ be a π-nef, numerically non-trivial line-bundle on the universal curve over $\overline{M}_{g,n}$. Then:

$$\mathcal{Z}(C) := \{ Z \subset C \mid \deg(\mathcal{L}|_Z) = 0 \}$$

defines an extremal assignment.

$$\begin{array}{c} \mathcal{C}^{\mathcal{L}} \\ \pi \downarrow \\ \overline{M}_{g,n} \end{array}$$

i.e. each face of the relative cone of curves $\overline{N}_1^+(\mathcal{C}/\overline{M}_{g,n}) \subset N_1(\mathcal{C}/\overline{M}_{g,n})$ gives rise to a stable modular compactification.

Examples:

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \overline{M}_2 \end{array}$$

$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{C}/\overline{M}_2) = \mathbb{Q}\{\omega_{\mathcal{C}/\overline{M}_2}\}$

Relative Nef-Cone: $\longrightarrow \omega_{\mathcal{C}/\overline{M}_2}$

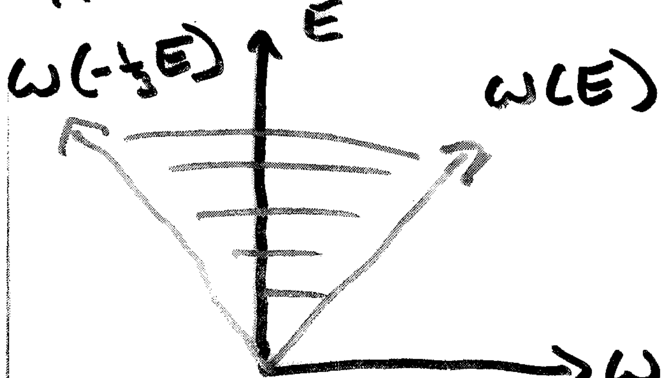
$\Rightarrow$ any π-nef line-bundle is π-ample so we get no new stability conditions.

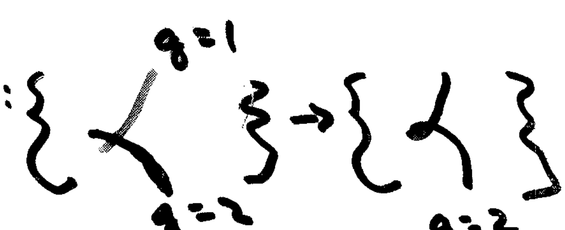
$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \overline{M}_3 \end{array}$$

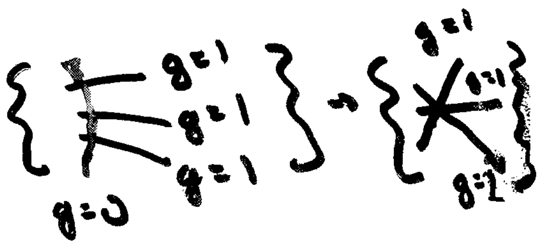
$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{C}/\overline{M}_3) = \mathbb{Q}\{\omega_{\mathcal{C}/\overline{M}_3}, E\}$

$E :=$ "Divisor of Elliptic Tails" $\{\{$ 

Relative Nef Cone:



$\omega(E)^{\perp}$: 

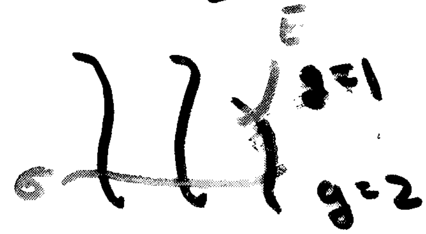
$\omega(-\frac{1}{3}E)^{\perp}$: 

$$\begin{array}{c}\mathcal{C}\\ \big\downarrow \pi\\ \overline{\mathcal{M}}_{2,2}\end{array}$$

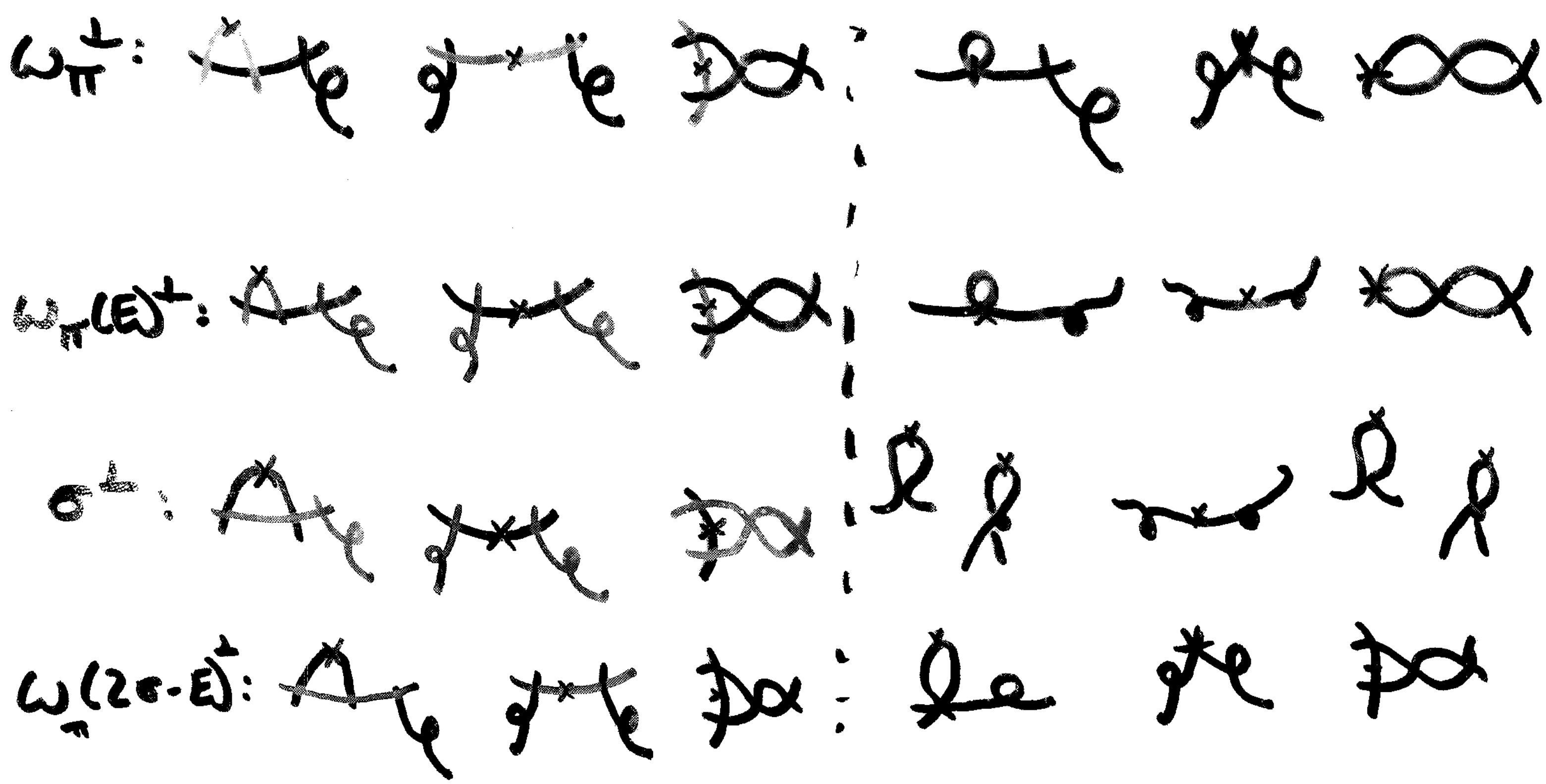
$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{C}/\overline{\mathcal{M}}_{2,1}) = \mathbb{Q}\{\omega_{\mathcal{C}/\overline{\mathcal{M}}_{2,1}}, E, \sigma\}$

E : divisor of unmarked elliptic tails

σ : image of section



Relative Nef Cone is generated by 4 extremal rays: ω, $\omega(E)$, σ, $\omega(2\sigma - E)$



Then we may consider codimension-2 faces

$\omega_\pi^\perp \cap \omega_\pi(E)^\perp$, $\omega_\pi(E)^\perp \cap \sigma^\perp$, $\sigma^\perp \cap \omega_\pi(2\sigma-E)^\perp$, $\omega_\pi(2\sigma-E)^\perp \cap \omega$

... For a total of 8 alternate stable modular compactifications.

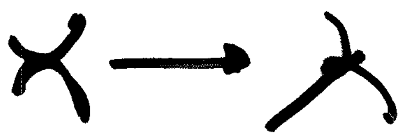
Does $\mathcal{Z}$-stability give a satisfactory theory of stability conditions for curves?

NO!!

<u>Corollary 1</u>: Suppose $\mathcal{X}$ is a stable modular compactification of $\mathcal{M}_{g,n}$. If one singularity of type-(h,m) arises in $\mathcal{X}$, then <u>all</u> singularities of type-(h,m) arise in $\mathcal{X}$.

e.g. There exists <u>no</u> stable mod. comp involving only nodes, cusps, and tacnodes.
$(y^2=x^2)$ $(y^2=x^3)$ $(y^2=x^4)$

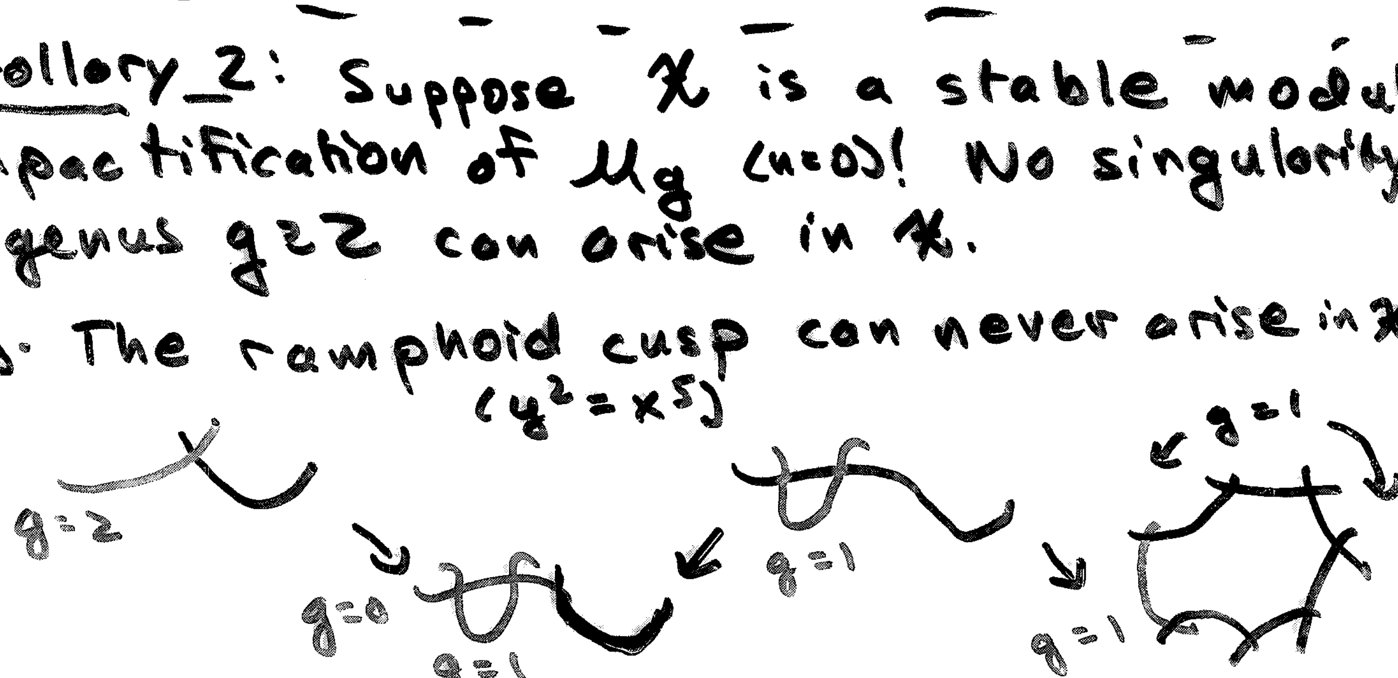
<u>Essential Problem</u>: Moduli stack of "crimping data" (van der Wyck) is not proper:



<u>Corollary 2</u>: Suppose $\mathcal{X}$ is a stable modular compactification of $\mathcal{M}_g$ $(n=0)$! No singularity of genus $g \geq 2$ can arise in $\mathcal{X}$.

e.g. The ramphoid cusp can never arise in $\mathcal{X}$.
$(y^2=x^5)$

<u>Pf</u>:



<u>Essential Problem</u>: Symmetry in dual graphs!

$g = 0$

In $g=0$, $\mathcal{Z}$-stability gives a perfect theory

Observation 1: If $(C,\{p_i\}_{i=1}^n)$ has $p_a=0$, then

$\mathrm{Aut}(C,\{p_i\}_{i=1}^n)$ proper $\Leftrightarrow$ $(C,\{p_i\}_{i=1}^n)$ is prestable

Corollary: Every modular comp. of $\mathcal{M}_{0,n}$ is stable

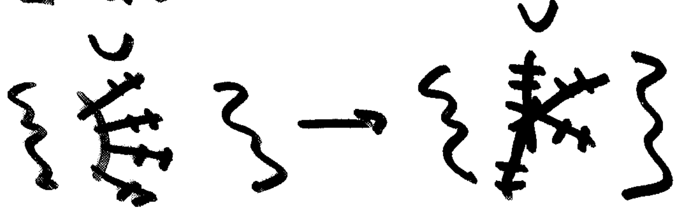
Observation 2: If C has $p_a=0$, then C is smoothable

Corollary: $\underline{\mathcal{U}_{0,n}} = \underline{\mathcal{V}_{0,n}}$, i.e. mod. comp. is open in $\mathcal{U}_{0,n}$

In sum, we have a complete classification of open proper substacks $\mathcal{X} \subset \mathcal{U}_{0,n}$.

We obtain regular birational contractions $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}(\mathcal{Z})$

e.g. $\overline{\mathcal{M}}_{0,8} \longrightarrow \overline{\mathcal{M}}_{0,8}(\mathcal{Z})$ small contraction of a curve on a 5-fold $\Rightarrow$ mod. space must be singular.



Questions: 1) When is $\overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ projective?

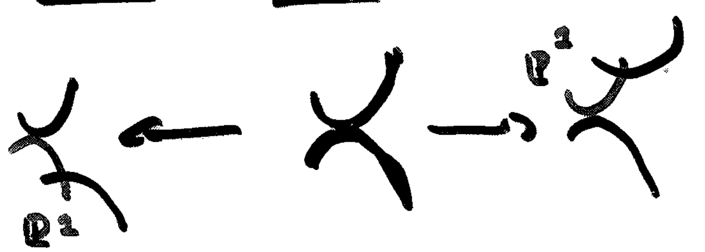
2) How much of nef-cone of $\overline{\mathcal{M}}_{0,n}$ is governed by these spaces?

(N.B. Castravet-Tevelev construct contractions $\overline{\mathcal{M}}_{0,n} \xrightarrow{\phi} X$ s.t. $\mathrm{Exc}(\phi) \cap \mathcal{M}_{0,n} \neq \emptyset$.)

$\underline{\underline{g=1}}$: Moduli of crimping data becomes a problem.

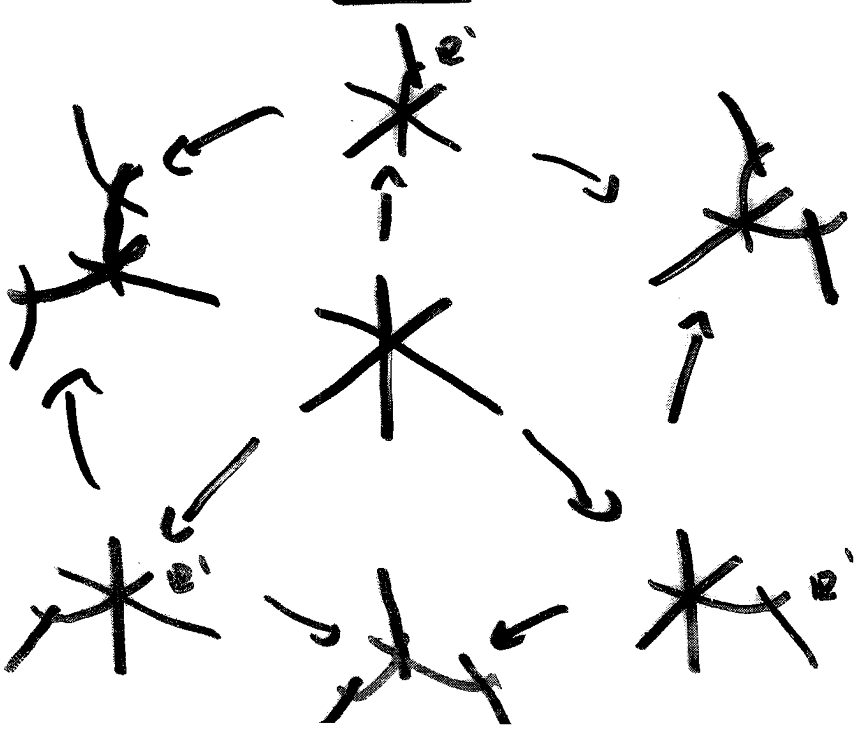
Moduli of Crimping Data for the Gorenstein elliptic m-fold point is $(\mathbb{C}^*)^{m-1}$.
There exists a <u>modular</u>, <u>equisingular</u> compactification $(\mathbb{C}^*)^{m-1} \subset \mathbb{P}^{m-1}$.

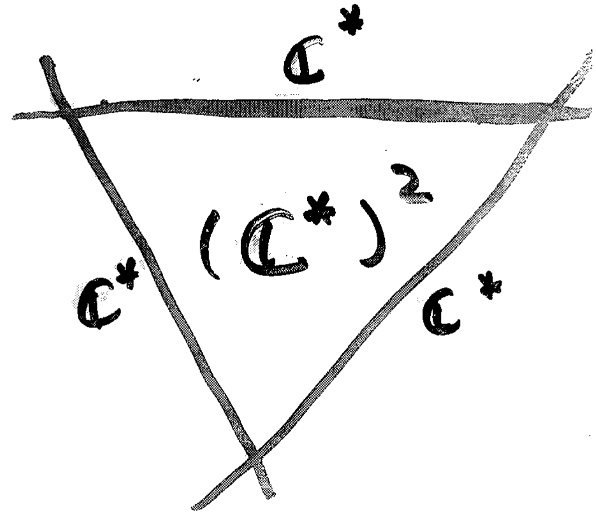
<u>E.g.</u>: <u>$m=2$</u>



This curve has neither automorphisms nor moduli of crimping data.

<u>$m=3$</u>





These equisingular compactifications lead to <u>semistable</u> modular compactifications of $M_{1,n}$:

<u>Defn</u>: We say that $p_a=1$ curve $(C,\{p_i\}_{i=1}^{n})$ is <u>m-stable</u> if

a) C has nodes, cusps, tacnodes, ..., Gorenstein elliptic m-fold points.

b) $\{p_i\}_{i=1}^{n}$ smooth, distinct.

c) If $E\subset C$ connected, $p_a=1$ subcurve, then
$$|E\cap E^c|>m.$$

d) $H^0(C,\Omega_C^\vee)=0$. (Equiv. up to $k-1$ branches of elliptic k-fold point may be semistable $\mathbb{P}^1$'s)

This gives rise to semistable comp's $\overline{M}_{1,n}(m)$ related by flips:

$$\overline{M}_{1,n} \longrightarrow \overline{M}_{1,n}(1) \dashrightarrow \overline{M}_{1,n}(2)\cdots$$

$g=1$

$$\overline{M}_{1,2}(1)\times\overline{M}_{0,n_1}\times\overline{M}_{0,n_2} \qquad \mathbb{P}(\psi_1\oplus\psi_2)$$

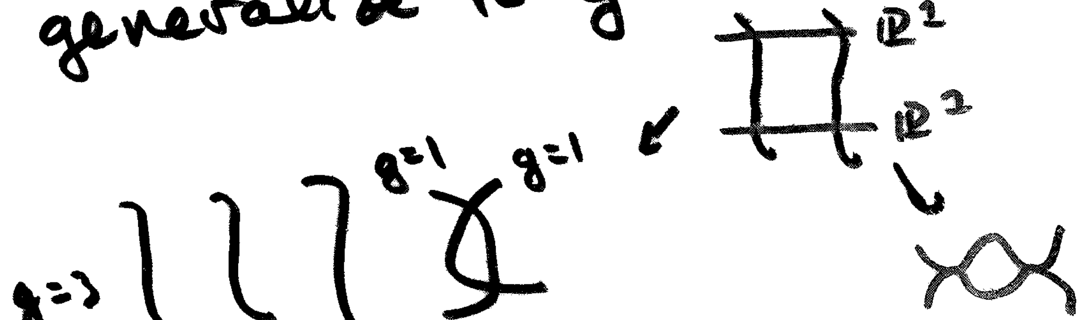
$$\searrow \qquad \swarrow$$

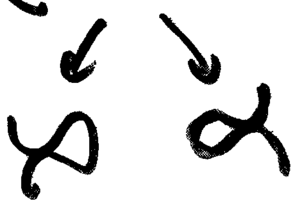
$$\overline{M}_{0,n_1}\times\overline{M}_{0,n_2}$$

$g \geq 2$: Now symmetry of dual graphs becomes a major problem

Question: Why doesn't the construction of a semistable modular compactification with nodes, cusps, and tacnodes generalize to $g \geq 2$.

Ex:





This looks like "canonical" limit but it has $\mathbb{C}^*$ automorphisms

In higher genus, rings of elliptic bridges continue to pose problems.

In fact, all these limits are strictly semistable points in

$\mathcal{H}ilb^{ss}_{2,g}$:= Semistable Locus of Hilb. Scheme of Bicanonical Curves for natural PGL action.

Corresponding Quotient space corresponds to a Mori Chamber in $\mathrm{Eff}(\overline{M}_g)$. (Hassett-Hyeon)

Suggests that for $g \geq 2$, we won't get a chamber decomposition using separated moduli functors.

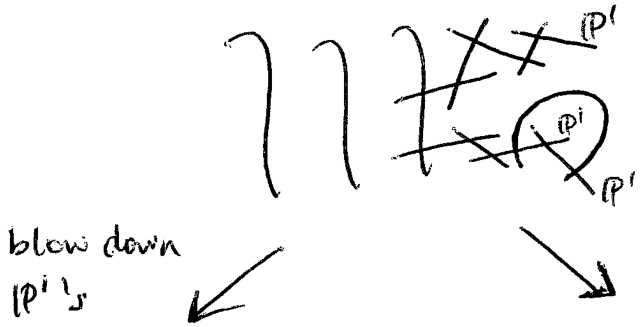
Suggestion: (w/ Alper)

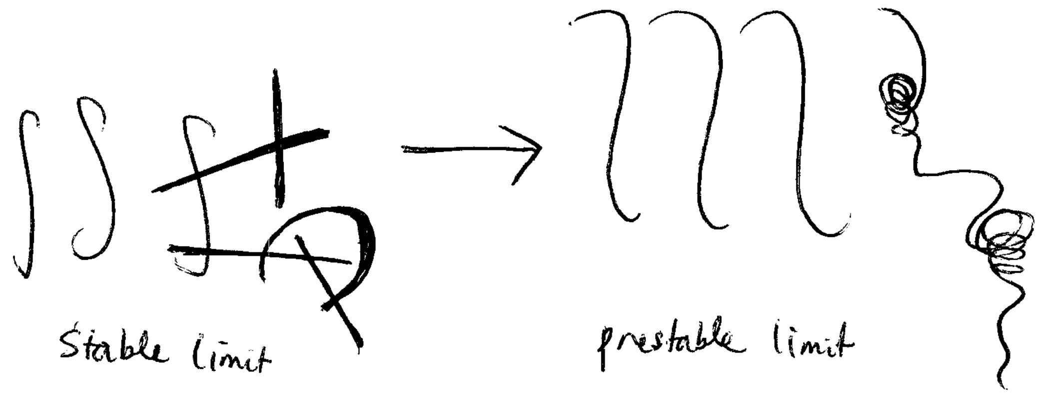
1) Give an abstract characterization of those mildly non-separated stacks which arise in such GIT constructions. (weakly proper Artin stacks ... familiar to those who study semistable sheaves)

2) Prove that such stacks admit a good moduli space (in sense of Alper)

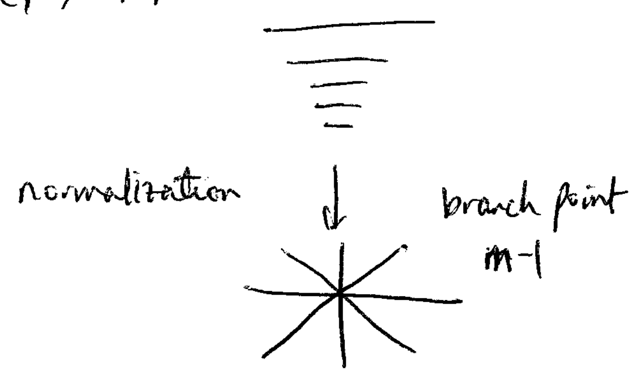
3) Give a combinatorial classification of open, weakly proper substacks $\mathcal{X} \subset \mathcal{U}_{g,n}$. (This may be impossible.)

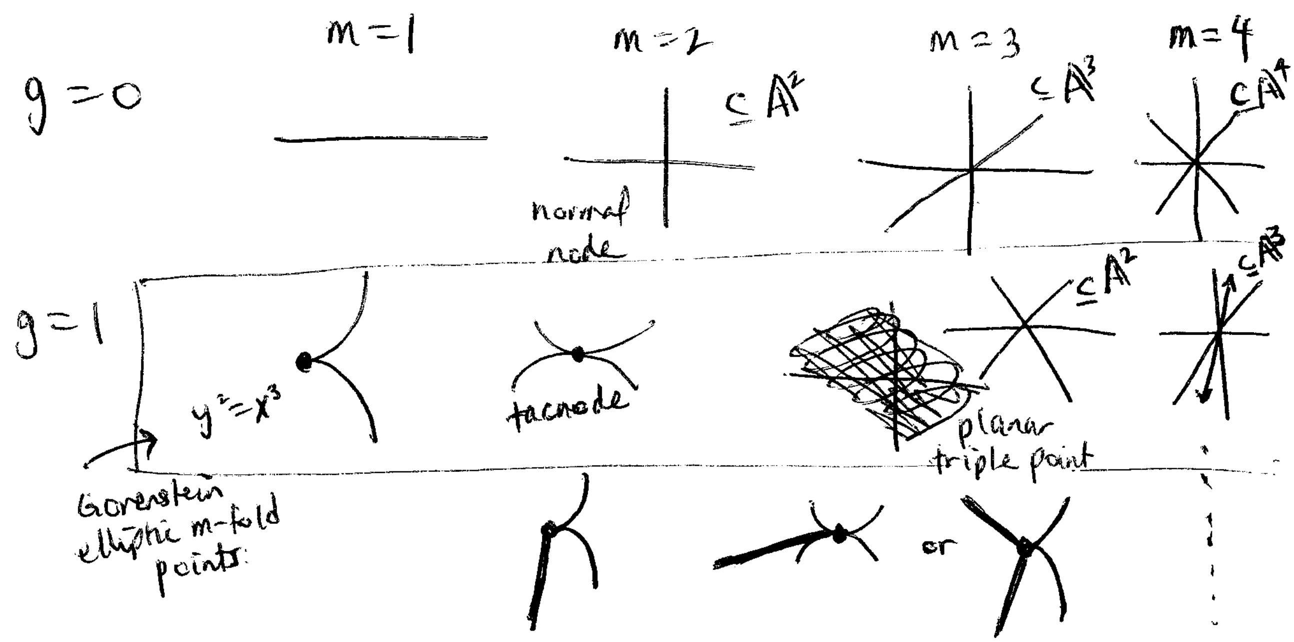
David Smyth
2-3PM
Feb 23, 2009

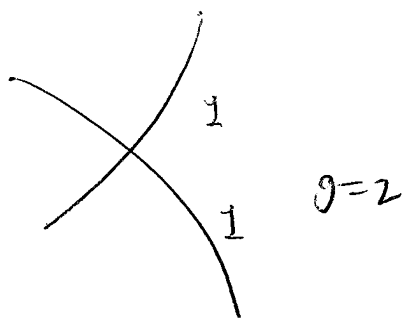




$P \in C$ singular point. Then we define
$g(P) = \delta(P) - m(P) + 1$

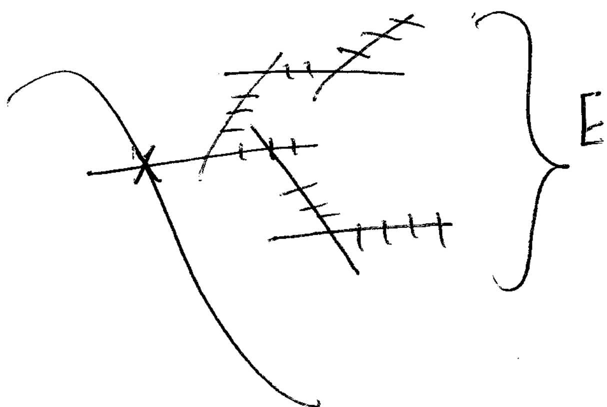




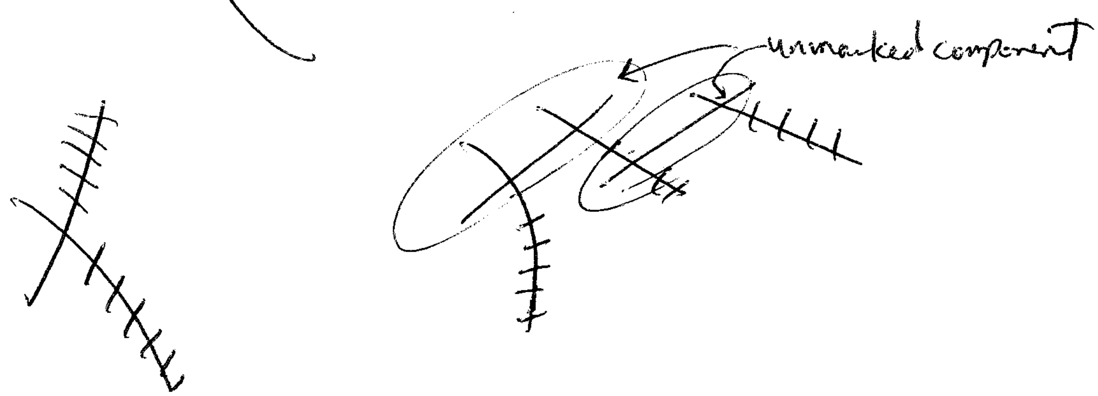


Fix $k \in \mathbb{Z}$.

$p_a = 0$ subcurve, $|E \cap E^c| = 1$, $\leq k$-marked points.



This is $\overline{M}_{g,A}$ with $A = \{\frac{1}{k}, \dots, \frac{1}{k}\}$



How do you make this into a classification?

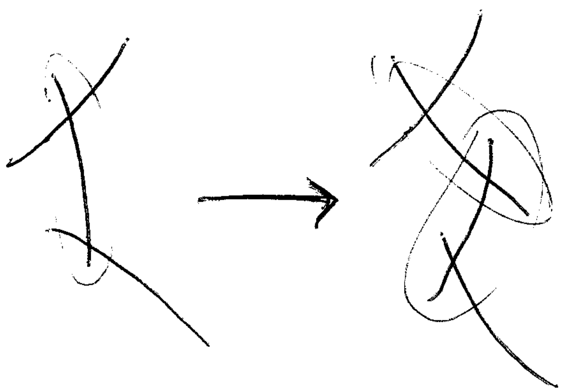
$$\begin{tikzcd}
\mathcal{C}^S \arrow[d] & \mathcal{C}^S \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[dl] & \mathcal{C} \arrow[d] \\
& T \arrow[dl] \arrow[drr] & & \\
\overline{M}_{g,n} & \subseteq U & \subseteq & \mathcal{X}
\end{tikzcd}$$

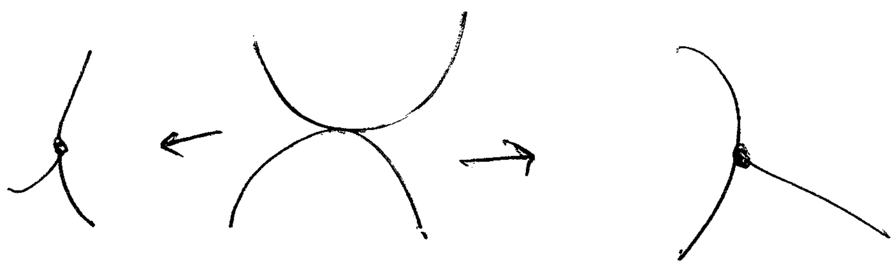
$$\begin{tikzcd}
\mathcal{C}^S \arrow[r, "\phi"] \arrow[d] & \mathcal{C} \arrow[dl] \\
s \in S \arrow[d, "\text{proper alteration}"] & \\
T \arrow[dl, no head] & \\
\overline{M}_{g,n} &
\end{tikzcd}$$

$$\mathrm{Exc}(\phi_S) \subseteq \mathcal{C}_S^S$$

$$\begin{tikzcd} & S_1, S_2, \ldots, S_R \\ \Delta \arrow[r] \arrow[ur, "\exists u_1, \ldots, u_R \text{ lifts}"] & \overline{M}_{g,n} \ni x \end{tikzcd}$$

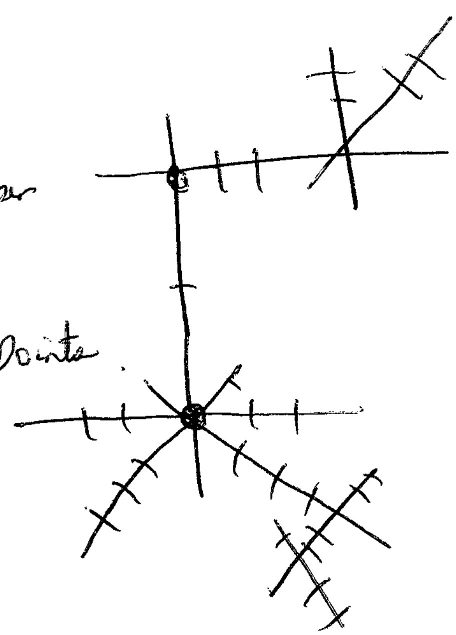
$\left.\begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_R \end{matrix}\right\}$ R connected components, fibers over x.





0 $\mathbb{C}^*$ ∞

have finite/proper aut. gp $\iff$ $\exists$ only 2 marked points.



family of $g=3$ curves