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Tom Bridgeland
Feb 24, 09 11AM–12PM

Hall algebras and curve-counting invariants

(1) X smooth projective CY_3 / $\mathbb{C}$

Take $\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$.

$\mathrm{Hilb}(\beta, n)$ parameterizes

$$\mathcal{O}_X \xrightarrow{f} E \qquad ch(E) = (\beta, n)$$

Behrend function $\nu : \mathrm{Hilb}(\beta, n) \longrightarrow \mathbb{Z}$

$$DT(\beta, n) = \int_{\mathrm{Hilb}(\beta,n)} \nu \, d\chi \in \mathbb{Z}$$

$$= \sum_{i \in \mathbb{Z}} \chi(\nu^{-1}(i))\, i$$

$$DT(\beta) = \sum_{n \in \mathbb{Z}} DT(\beta, n)\, t^n$$

Laurent series

$P\text{-}\mathrm{Hilb}(\beta, n)$ parameterizes

$$\mathcal{O}_X \xrightarrow{f} E \qquad ch(E) = (\beta, n)$$

E pure of dimension 1, coker(f) dimension 0.

$$PT(\beta,n) = \int_{P\text{-}Hilb(\beta,n)} \nu \, d\chi \in \mathbb{Z}$$

$$PT(\beta) = \sum_{n\in\mathbb{Z}} PT(\beta,n)\, t^n$$

<u>Conjecture</u> (P–T)

(a) $DT(\beta) = PT(\beta) \cdot DT(0)$

(b) $PT(\beta)$ Laurent expansion of a rational function in t invariant under $t \leftrightarrow 1/t$.

<u>Claim</u>: These statements follow from work of Joyce on Hall algebras. Toda has different proof for invariants defined without ν.

Set $\mathcal{A} = Coh(X)$, $\mathcal{D} = D^b(\mathcal{A})$

$\exists$ torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$

$\mathcal{T} = \{E \in \mathcal{A} : \dim supp(E) = 0\}$

$\mathcal{F} = \{E \in \mathcal{A} : Hom_{\mathcal{A}}(T, E) = 0 \ \forall T \in \mathcal{T}\}$

$$\overbrace{\boxed{\mathcal{T}} \boxed{\quad\mathcal{F}\quad}}^{\mathcal{A}} \;\Big\|\; \overbrace{\boxed{\mathcal{T}[-1]} \boxed{\mathcal{F}[-1]}}^{\mathcal{A}[-1]}$$

$$\underbrace{\mathcal{F} \;\|\; \mathcal{T}[-1]}_{\mathcal{B}}$$

Abelian category

$$\mathcal{B} = \left\{ E \in D \;\middle|\; H^i(E) = \begin{cases} \in \mathcal{F} & i=0 \\ \in \mathcal{T} & i=1 \\ 0 & \text{o/w} \end{cases} \right\}$$

<u>Lemma</u>: P-Hilb(β, n) parametrizes quotients $\mathcal{O}_X \twoheadrightarrow E$ in $\mathcal{B}$ with $[E] = (\beta, n)$.

<u>Proof</u>: Given short exact sequence

$$I \to \mathcal{O}_X \xrightarrow{f} E \quad \text{in } \mathcal{B}$$

get
$$H^0(I) \to \mathcal{O}_X \xrightarrow{f} \underset{\substack{\| \\ E \\ \cap \\ \mathcal{F}}}{H^0(E)} \to \underset{\substack{\cap \\ \mathcal{T}}}{H^1(T)}$$

Converse just as easy. $\square$

② ~~...~~ <u>Joyce's stacky Hall algebra</u>

$\mathcal{M}$ = ~~...~~ stack of objects of $\mathcal{A}$

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$\mathcal{M}^{(2)} =$ stack of ~~objects~~ short exact sequences in $\mathcal{A}$

$$\mathcal{M}^{(2)} \xrightarrow{\pi_i} \mathcal{M} \qquad i=1,2,3$$

$$(0 \to A_1 \to A_2 \to A_3 \to 0) \mapsto A_i$$

$$H(\mathcal{A}) = K(St/\mathcal{M}) = \bigoplus \mathbb{Z}[\mathcal{X} \xrightarrow{f} \mathcal{M}]$$

with $\mathcal{X}$ stack of finite type

f representable

with relations

$$[\mathcal{X} \xrightarrow{f} \mathcal{M}] = [\mathcal{Y} \xrightarrow{f} \mathcal{M}] + [\mathcal{X} \setminus \mathcal{Y} \xrightarrow{f} \mathcal{M}]$$

for $\mathcal{Y} \subset \mathcal{X}$ a closed substack

Convolution product

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{M}] * [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{M}] = [\mathcal{X}_3 \xrightarrow{\pi_2 \circ g} \mathcal{M}]$$

$$\begin{tikzcd}
\mathcal{X}_3 \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{(2)} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1, \pi_3)"] & \mathcal{M} \\
\mathcal{X}_1 \times \mathcal{X}_2 \arrow[r, "(f_1, f_2)"'] & \mathcal{M} \times \mathcal{M} &
\end{tikzcd}$$

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Associative because

$$\begin{tikzcd} A\subset B\subset E \arrow[d, "I"] & \mathcal{M}^{(3)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}^{(2)} \arrow[d, "\pi_1"] & B\subset E \\ A\subset B & \mathcal{M}^{(2)} \arrow[r, "\pi_2"'] & \mathcal{M} & B \end{tikzcd}$$

$$\mathcal{M} = \coprod_{\alpha\in K(A)} \mathcal{M}_\alpha \implies H(A) = \bigoplus_{\alpha\in K(A)} H_\alpha(A)$$

$[\mathcal{M}_0 = \mathrm{Spec}(\mathbb{C}) \subset \mathcal{M}]$ is unit.

Define a $\mathbb{C}(q)$-algebra

$$\mathbb{C}_q[K(A)] = \bigoplus_{\alpha\in K(A)} \mathbb{C}(q)\, x^\alpha$$

$$x^\alpha * x^\beta = q^{\chi(\alpha,\beta)}\, x^{\alpha+\beta}$$

<u>Conjecture</u>: $\exists$ ring HM

$$I : H(A) \longrightarrow \mathbb{C}_q[K(A)]$$

s.t. $$\lim_{q\to 1} I([X \xrightarrow{f} \mathcal{M}_\alpha]) = \int_X f^*(\nu)\, dx \cdot x^\alpha$$

if X is a scheme.

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Proved by Kontsevich-Soibelman if X has "orientation data". $\exists$ alternative technology of Joyce less easy to state but on a firmer footing.

③ <u>Proof of DT/PT</u>

Take now $\mathcal{A} = \mathrm{Coh}_{\leq 1}(X)$, $\mathcal{M} =$ stack of objects of $\mathcal{A}$.

Note that $\chi(-,-) \equiv 0$ so $\mathbb{C}_q[K(\mathcal{A})]$ is commutative.

Introduce some elements of $\widehat{H(\mathcal{A})}$.

$$\text{P-Hilb} : \left[\coprod_{(\beta,n)} \text{P-Hilb}(\beta,n) \xrightarrow{r} \mathcal{M} \right]$$

$$\text{Hilb} : \left[\coprod_{(\beta,n)} \text{Hilb}(\beta,n) \xrightarrow{q} \mathcal{M} \right]$$

$$\text{Hilb}_0 : \left[\coprod_{n} \text{Hilb}(0,n) \xrightarrow{q} \mathcal{M} \right]$$

We must prove

$$I(\text{Hilb}) = I(\text{Hilb}_0) \cdot I(\text{P-Hilb})$$

We can then apply a result of P-T

$$\mathrm{Hilb}(\beta, n) \xrightarrow{i} \mathcal{M}$$

$$i^*(\nu_{\mathcal{M}}) = (-1)^n \, \nu_{\mathrm{Hilb}(\beta,n)}$$

and similarly for $P-\mathrm{Hilb}(\beta, n)$

Also introduce

$$\mathbb{1}_{\mathcal{A}} : \quad [\mathcal{M} \xrightarrow{id} \mathcal{M}]$$

$$\mathbb{1}_{\mathcal{T}} : \quad [\mathrm{Obj}\,\mathcal{T} \hookrightarrow \mathcal{M}]$$

$$\mathbb{1}_{\mathcal{F}} : \quad [\mathrm{Obj}\,\mathcal{F} \hookrightarrow \mathcal{M}]$$

$$\mathbb{1}^{\mathcal{O}}_{\mathcal{A}} : \quad [\mathrm{Map}^{\mathcal{O}} \longrightarrow \mathcal{M}]$$

and similarly $\mathbb{1}^{\mathcal{O}}_{\mathcal{T}}$, $\mathbb{1}^{\mathcal{O}}_{\mathcal{F}}$

Here $\mathrm{Map}^{\mathcal{O}}$ parameterizes

$E \in \mathcal{A}$ together with $\mathcal{O}_X \xrightarrow{f} E$.

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Now: $I(\mathrm{Hilb}) \overset{?}{=} I(\mathrm{Hilb}_0)\cdot I(\text{P-Hilb})$

$$\begin{array}{ccc} \| \text{A} & \| \text{B} & \| \text{C} \end{array}$$

$$I(1_A^{\mathcal{O}} * 1^{-1}) \overset{?}{=} I(1_\uparrow^{\mathcal{O}} * 1_\uparrow^{-1}) \cdot I(1_y^{\mathcal{O}} * 1_y^{-1})$$

$$\| \text{D} \qquad \| \text{E}$$

$$I(1_\uparrow^{\mathcal{O}} * 1_y^{\mathcal{O}} * 1_y^{-1} * 1_\uparrow^{-1})$$

Reordering gives result

For A:

IM on geometric point,

$$\begin{tikzcd} & \mathrm{Map}^{\mathcal{O}} \arrow[rr, bend left, "\text{E}"] & & \\ Z \arrow[ur] \arrow[r] \arrow[d] & M^{(2)} \arrow[rr, "\text{ACF}"] \arrow[d, "\pi_1"] & & M \\ \mathrm{Hilb} \arrow[r] & M & \text{A} & \end{tikzcd}$$

$\mathcal{O}_x \to E$

$\mathcal{O}_x \to A$

So $\mathrm{Hilb} * 1_A = 1_A^{\mathcal{O}}$.

B exactly same.

For C may help to observe

$$1_B^{\mathcal{O}} * 1_B^{-1} = 1_y^{\mathcal{O}} * \underbrace{1_{\uparrow[1]}^{\mathcal{O}} * 1_{\uparrow[1]}^{-1}}_{\overset{\|}{1}} * 1_y^{-1}$$

but

$\nexists\ \mathcal{O}_x \to T[1]$

Hall algebras and Curve Counting

Tom Bridgeland
Feb 24, 2009
11AM – 12PM

(1.) Curve-counting invariants

X smooth, projective CY_3 / $\mathbb{C}$
$\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$

$\mathrm{Hilb}(\beta, n)$ parametrizes $\mathcal{O}_X \twoheadrightarrow E$ coherent sheaf

$$[E] = (\beta, n)$$

Behrend function : $\nu : \mathrm{Hilb}(\beta, n) \to \mathbb{Z}$

$$DT(\beta, n) = \int_{\mathrm{Hilb}(\beta,n)} \nu \, d\chi = \sum_{i \in \mathbb{Z}} \chi(\nu^{-1}(i))\, i$$

P-$\mathrm{Hilb}(\beta, n)$ (Preverse Hilb Scheme) param. $\mathcal{O}_X \xrightarrow{f} E$ (nearly a surj.) $[E] = (\beta, n)$

s.t. E is pure dim 1, $\mathrm{coker}(f)$ is supported in dim 0.

$$PT(\beta, n) = \int_{P\text{-}\mathrm{Hilb}(\beta,n)} \nu \, d\chi \in \mathbb{Z}$$

$$DT(\beta) = \sum_{n \in \mathbb{Z}} DT(\beta, n) t^n$$

$$PT(\beta) = \sum_{n \in \mathbb{Z}} PT(\beta, n) t^n$$

By geom. vanishing, Laurent series

Conjecture (P–T)

(a.) $DT(\beta) = PT(\beta) \cdot DT(0)$

$$DT(0) = \sum_{n \in \mathbb{Z}_{\geq 0}} DT(0,n)\, t^n$$

(b.) $PT(\beta)$ is the Laurent-expansion of a rat'l fn. of t invariant under $t \longleftrightarrow t^{-1}$.

Claim These statements follow from Joyce's recent results.

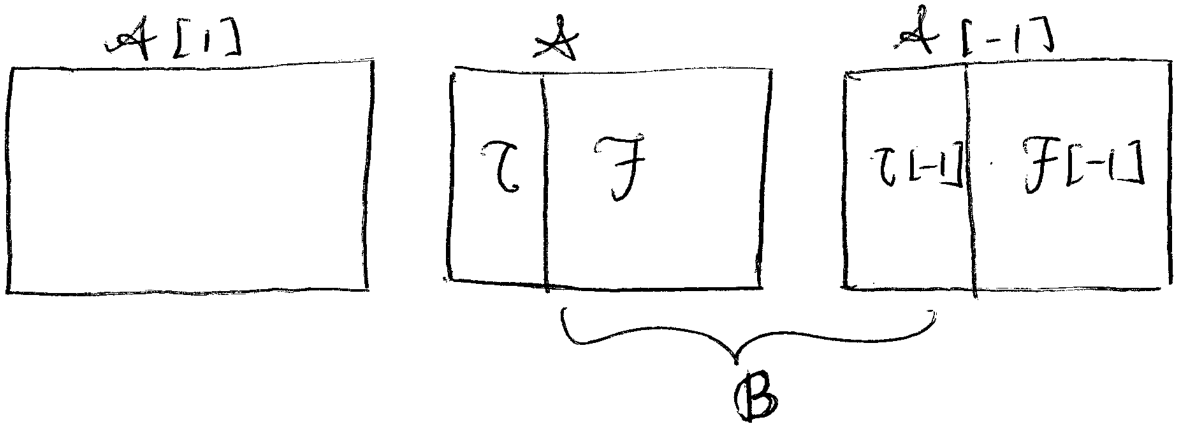
> Y. Toda also has an approach to such results using Joyce's work.

Set $\mathcal{A} = \mathrm{Coh}(X)$, $D = D^b\mathrm{Coh}(X)$.

Define a torsion pair $(\mathcal{T}, \mathcal{F}) \subseteq \mathcal{A}$

$\mathcal{T} = \{E \in \mathcal{A} : \mathrm{supp}(E) \text{ of dim } 0\}$

$\mathcal{F} = \{E \in \mathcal{A} : \mathrm{Hom}_{\mathcal{A}}(T, E) = 0 \ \forall T \in \mathcal{T}\}$



Define $\mathcal{B} = \left\{ E \in D : H^i(E) = \begin{cases} \in \mathcal{F} & i = 0 \\ \in \mathcal{T} & i = 1 \\ 0 & \text{otherwise} \end{cases} \right\}$

Lemma. $P\text{-}\mathrm{Hilb}(\beta, n)$ parametrizes quotients $\mathcal{O}_X \twoheadrightarrow E$ in $\mathcal{B}$, with $[E] = (\beta, n)$.

<u>Proof</u> Given a SES
$$I \to \mathcal{O}_X \to E \quad \text{in } \mathcal{B}.$$
Then take LES of cohomology
$$\begin{array}{ccccccccc} 0 \to & H^0(I) & \to & \mathcal{O}_X & \to & H^0(E) & \to & H^1(I) & \to 0 \\ &&&&& \parallel && \cap & \\ &&&&& E && \mathcal{T} & \\ &&&&& \cap &&& \\ &&&&& \mathcal{F} &&& \end{array}$$
The other way around equally arg.

$\boxtimes$

② Joyce's stacky Hall algebra

$\mathcal{M}$ stack of objects of $\mathcal{A}$, $\mathcal{M}^{(2)}$ = stack of SES's in $\mathcal{A}$

$\exists$ maps $\pi_i : \mathcal{M}^{(2)} \to \mathcal{M} \quad i = 1,2,3$
$$(0 \to A_1 \to A_2 \to A_3 \to 0) \longmapsto A_i$$
$$H(\mathcal{A}) = K(\mathrm{st}/\mathcal{M}) = \bigoplus \mathbb{C}[\mathcal{X} \xrightarrow{f} \mathcal{M}]$$
where $\mathcal{X}$ stack of finite type,
f representable map (inj. on stabilizer groups),
up to relations
$$[\mathcal{X} \xrightarrow{f} \mathcal{M}] = [\mathcal{Y} \xrightarrow{f} \mathcal{M}] + [\mathcal{X} \setminus \mathcal{Y} \xrightarrow{f} \mathcal{M}]$$
for $\mathcal{Y} \subseteq \mathcal{X}$ closed substack.

Convolution product:
$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{M}] * [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{M}] = [\mathcal{Z} \xrightarrow{\pi_2 \circ g} \mathcal{M}]$$

$\longrightarrow$

where

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}^{(2)} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1, \pi_3)"] & \mathcal{M} \\ \mathcal{X}_1 \times \mathcal{X}_2 \arrow[r, "(f_1, f_2)"] & \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

univ. extension

fiber over (E_1, E_2) is $\left[Ext^1(E_2, E_1) / Hom(E_2, E_1) \right]$

Associative algebra because

$$\begin{tikzcd} \mathcal{M}^{(3)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}^{(2)} \arrow[d, "\pi_1"] \\ \mathcal{M}^{(2)} \arrow[r, "\pi_2"] & \mathcal{M} \end{tikzcd}$$

$A \subseteq B \subseteq E$ 2-step filtration

$B \subseteq E$

$A \subseteq B$

B

$$\mathcal{M} = \coprod \mathcal{M}_\alpha \Rightarrow H(\mathcal{A}) \text{ is } K(\mathcal{A})$$

graded

$\alpha \in H^*(X, \mathbb{Q})$
or $\alpha \in K(\mathcal{A})$

Identity is $[\, \mathcal{M}_0 = Spec(\mathbb{C}) \subseteq \mathcal{M} \,]$

zero objects

Define a $\mathbb{C}(q)$-algebra

$$\mathbb{C}_q[K(\mathcal{A})] = \bigoplus_{\alpha \in K(\mathcal{A})} \mathbb{C}(q)\, X^\alpha$$

with product $X^\alpha * X^\beta = q^{\chi(\alpha, \beta)} X^{\alpha+\beta}$,
where $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is the Euler form,

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} Ext^i(E, F).$$

$CY_3 \Rightarrow \chi[-,-]$ skew symmetric.

<u>Conjecture</u>: $\exists$ ring HM, Lawrent ring.

$I : H(\mathcal{A}) \to \mathbb{C}_q[K(\mathcal{A})]$ commutative when $q=1$.

such that

$$\lim_{q\to 1} I([\underset{\text{scheme}}{Y} \xrightarrow{f} \mathcal{M}_\alpha]) = \left(\int_Y f^*(\nu_M)\, dx\right) x^\alpha$$

$\exists$ well-behaving limit if Y scheme.

$\mathcal{M} \xrightarrow{\nu} \mathbb{Z}$

Possibly proved by Kontsevich-Soibelman when X has "orientation data."

$\exists$ Alternative technology of Joyce, not so easy to state but on a firm footing.

<u>DT/PT</u>: Now take $\mathcal{A} = \mathrm{Coh}_{\leq 1}(X)$,
$\mathcal{M}$ = stack of objects of $\mathcal{A}$

Note that $\chi(-,-) \equiv 0$.
So this ring $\mathbb{C}_q[K(\mathcal{A})]$ is <u>commutative</u>.

Introduce some elements of $\widehat{H(\mathcal{A})}$

not of fin type, so not elmts of Hall algebra.

Hilb $\quad [\coprod_{(\beta,n)} \mathrm{Hilb}(\beta,n) \xrightarrow{q} \mathcal{M}]$

P-Hilb $\quad [\coprod_{(\beta,n)} \text{P-Hilb}(\beta,n) \xrightarrow{q} \mathcal{M}]$

Hilb_0 $\quad [\coprod \mathrm{Hilb}(0,n) \xrightarrow{q} \mathcal{M}]$

What we have to prove is
$I(\mathrm{Hilb}) = I(\mathrm{Hilb}_0) \cdot I(\rho\text{-}\mathrm{Hilb})$.

$$\int_{\mathrm{Hilb}(\beta,n)} \nu \, d\chi = (-1)^n \int_{\mathrm{Hilb}(\beta,n)} q^* \nu \, d\chi$$

$$\nu_{\mathrm{Hilb}(\beta,n)} \overset{?}{=} (-1)^n \, q^* \nu_{\mathcal{M}}$$

Also introduce other elements:

$\mathbb{1}_{\mathcal{M}}$ $\quad [\mathcal{M} \xrightarrow{id} \mathcal{M}]$

$\mathbb{1}_{\mathcal{T}}$ $\quad [\mathrm{Obj}(\mathcal{T}) \subseteq \mathcal{M}]$ open substacks. dim 0 sheaves

$\mathbb{1}_{\mathcal{F}}$ $\quad [\mathrm{obj}(\mathcal{F}) \subseteq \mathcal{M}]$ pure dim 1 sheaves.

$\mathbb{1}^{\mathcal{O}}_{\mathcal{M}}$ $\quad [\mathrm{Map}^{\mathcal{O}} \to \mathcal{M}]$ parametrizing $E \in \mathcal{A}$ together with $\mathcal{O}_X \to E$.
$H^0(X, E) \longmapsto \overset{\cup}{E}$

$\mathbb{1}^{\mathcal{O}}_{\mathcal{T}}$

$\mathbb{1}^{\mathcal{O}}_{\mathcal{F}}$

$$I(\mathrm{Hilb}) \overset{?}{=} I(\mathrm{Hilb}_0)\cdot I(P\text{-}\mathrm{Hilb})$$

in Hall$_{alg}$. $\| (A)$ $\quad \| (B)$ $\quad \| (C)$

$$I(\mathbb{1}_{\mathcal{M}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{M}}^{-1}) \overset{?}{=} \underline{I(\mathbb{1}_{\mathcal{T}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{T}}^{-1}) \cdot I(\mathbb{1}_{\mathcal{F}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{F}}^{-1})}$$

$\| (D)$ $\quad \| (E)$ $\quad \|$

$$I\left((\mathbb{1}_{\mathcal{T}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{F}}^{\mathcal{O}}) * (\mathbb{1}_{\mathcal{F}}^{-1} * \mathbb{1}_{\mathcal{T}}^{-1})\right)$$

(A): $\mathbb{1}_{\mathcal{M}}^{\mathcal{O}} = \mathrm{Hilb} * \mathbb{1}_{\mathcal{M}}$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}^{(2)} \arrow[r, "\pi_2"] \arrow[d, "(\pi_1, \pi_3)"] & \mathcal{M} \\ \mathrm{Hilb} \times \mathcal{M} \arrow[r, "(q, id)"'] & \mathcal{M} \times \mathcal{M} & \end{tikzcd}$$

$$(Z \to \mathcal{M}) = (\mathrm{Map}^{\mathcal{O}} \to \mathcal{M})$$

$$\begin{tikzcd} & \mathrm{Map}\,\mathcal{O} & \\ Z \arrow[ur] \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}^{(2)} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathcal{M} \\ \mathrm{Hilb} \arrow[r, "q"'] & \mathcal{M} & \end{tikzcd}$$

Pts. of Z param. $\mathcal{O}_X \to E$

param $A \subseteq E \to B$

IM on geometric points.

$\mathcal{O}_X \to A$

(A)

(c) $p\text{-Hilb} = \mathbb{1}_{\mathcal{B}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{B}}^{-1}$

$\mathbb{1}_{\mathcal{M}} = \underset{\text{has torsion}}{\mathbb{1}_{\mathcal{T}}} * \underset{\text{torsion free}}{\mathbb{1}_{\mathcal{F}}}$

$\mathbb{1}_{\mathcal{B}} = \mathbb{1}_{\mathcal{F}} * \mathbb{1}_{\mathcal{T}[-1]}$

$\mathbb{1}_{\mathcal{B}}^{\mathcal{O}} = \mathbb{1}_{\mathcal{F}}^{\mathcal{O}} * \mathbb{1}_{\mathcal{T}[-1]}^{\mathcal{O}}$ ← has no global sections.

$\mathcal{O} \not\to \mathcal{T}[-1]$

So

$$p\text{-Hilb} = \mathbb{1}_{\mathcal{F}}^{\mathcal{O}} * \cancel{\mathbb{1}_{\mathcal{T}[-1]}^{\mathcal{O}} * \mathbb{1}_{\mathcal{T}[-1]}^{-1}} * \mathbb{1}_{\mathcal{F}}^{-1}.$$