A theory of generalized Donaldson–Thomas invariants

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February, 2009

joint with Yinan Song
based on
arXiv:0810.5645,
arXiv:0901.2872,
and in preparation

These slides available at

`www.maths.ox.ac.uk/~joyce/talks.html`

1. Introduction

Let X be a Calabi–Yau 3-fold, and $\mathcal{A} = \mathrm{coh}(X)$ the abelian category of *coherent sheaves* on X. Write $K(\mathcal{A})$ for the *numerical Grothendieck group* of $\mathcal{A}$. If E is a coherent sheaf on X, write $[E]$ for its class in $K(\mathcal{A})$. The *Chern character* $\mathrm{ch}(E)$ lies in $H^{\mathrm{even}}(X;\mathbb{Q})$. It descends to a group morphism $\mathrm{ch} : K(\mathcal{A}) \to H^{\mathrm{even}}(X;\mathbb{Q})$. So $K(\mathcal{A})$ is a finite rank lattice $\mathbb{Z}^n$, a subgroup of $H^{\mathrm{even}}(X;\mathbb{Q})$. The *Euler form* is $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$, antisymmetric and biadditive. Using Serre duality gives

$$\begin{aligned} &\dim \mathrm{Hom}(E,F) - \dim \mathrm{Ext}^1(E,F) \\ &- \dim \mathrm{Hom}(F,E) + \dim \mathrm{Ext}^1(F,E) = \chi([E],[F]). \end{aligned} \tag{1}$$

Choose an ample line bundle $\mathcal{L}$ on X. This induces a notion of *Gieseker stability* on $\mathcal{A} = \mathrm{coh}(X)$. Write τ for the stability condition coming from $\mathcal{L}$. It depends on $\mathcal{L}$, so a different ample line bundle $\tilde{\mathcal{L}}$ induces a different stability condition $\tilde{\tau}$.

Given $\alpha \in K(\mathcal{A})$, we can form the moduli spaces $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau), \mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ of τ-(semi)stable sheaves E in $\mathcal{A}$ with $[E] = \alpha$ in $K(\mathcal{A})$. We can regard these as *schemes*, with points of $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ being S-equivalence classes of τ-semistable sheaves, rather than isomorphism classes. Alternatively, we can regard them as *Artin stacks*, as open constructible subsets in the moduli stack $\mathfrak{M}$ of all coherent sheaves.

Donaldson–Thomas invariants $\mathrm{DT}^\alpha(\tau)$ are integer-valued invariants 'counting' τ-(semi) stable sheaves in class $\alpha \in K(\mathcal{A})$. They are defined only in the case when $\mathcal{M}_{\mathrm{st}}^\alpha(\tau) = \mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$, that is, when there are no strictly semistable sheaves in class α.

The interesting property of Donaldson–Thomas invariants is that they are unchanged by continuous deformations of the underlying Calabi–Yau 3-fold X, that is, they are independent of the complex structure J of X up to deformation. This is a strong statement, as deforming X can change $\mathcal{A}$ and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ radically.

Until now, it was not known how $\mathrm{DT}^\alpha(\tau)$ depends on τ, that is, on the choice of ample line bundle L.

Kai Behrend showed that $\mathrm{DT}^\alpha(\tau)$ can be written as a *weighted Euler characteristic*

$$\mathrm{DT}^\alpha(\tau)=\int_{\mathcal{M}_{\mathrm{st}}^\alpha(\tau)}\nu\,\mathrm{d}\chi, \qquad (2)$$

where ν is the 'microlocal function', a $\mathbb{Z}$-valued constructible function on $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ depending only on the scheme structure of $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. We think of ν as a *multiplicity function*. If $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is a k-fold point $\mathrm{Spec}\,\mathbb{C}[z]/(z^k)$ then $\nu\equiv k$. If $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is smooth of dimension d then $\nu\equiv(-1)^d$.

In a series of previous papers, I defined a different set of invariants $J^\alpha(\tau) \in \mathbb{Q}$ 'counting' τ-semistable sheaves in class α. They are defined for all $\alpha \in K(\mathcal{A})$, including classes with strictly semistables. If $\mathcal{M}_{\mathrm{st}}^\alpha(\tau) = \mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$ then $J^\alpha(\tau)$ is the (unweighted) Euler characteristic $\chi\bigl(\mathcal{M}_{\mathrm{st}}^\alpha(\tau)\bigr) \in \mathbb{Z}$.
The important property of the $J^\alpha(\tau)$ is that their transformation law under change of stability condition is known: we can write $J^\alpha(\tilde\tau)$ as a sum of products of $J^\beta(\tau)$, with combinatorial coefficients.
However, the $J^\alpha(\tau)$ are not invariant under deformations of the underlying Calabi-Yau 3-fold. This is because they do not count points in $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ with multiplicity, so a k-fold point $\operatorname{Spec}\mathbb{C}[z]/(z^k)$ in $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ would contribute 1 to $J^\alpha(\tau)$, for instance.

The goal of the project

We will define a family of *generalized D–T invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ defined for all $\alpha \in K(\mathcal{A})$, combining the good properties of both the D–T invariants $\mathrm{DT}^\alpha(\tau)$, and my invariants $J^\alpha(\tau)$. That is:

- $\bar{DT}^\alpha(\tau)$ is unchanged by deformations of the underlying Calabi–Yau 3-fold.
- If $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)=\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ then $\bar{DT}^\alpha(\tau)=\mathrm{DT}^\alpha(\tau)$.
- The $\bar{DT}^\alpha(\tau)$ transform according to a known transformation law under change of stability condition. (As for the $J^\alpha(\tau)$, but with sign changes).

The general method is fairly obvious: we define $\bar{DT}^\alpha(\tau)$ by inserting Behrend's microlocal function ν as a weight in the definition of my $J^\alpha(\tau)$, so that the $\bar{DT}^\alpha(\tau)$ count sheaves with the correct multiplicity. But the details are complex.

2. A sketch of the $J^\alpha(\tau)$ set up

The invariants $J^\alpha(\tau)$, and other invariants, are defined and studied in 7 papers (the main four on 'Configurations in abelian categories') totalling 412 journal pages. Here is an oversimplified sketch:

Write $\mathcal{M}_\mathcal{A}$ for the moduli stack of sheaves in $\mathcal{A}$, an Artin stack. We define a $\mathbb{Q}$-vector space of 'stack functions' $\mathrm{SF}(\mathcal{M}_\mathcal{A})$, a generalization of $\mathbb{Q}$-valued constructible functions on $\mathcal{M}_\mathcal{A}$. Loosely, $\mathrm{SF}(\mathcal{M}_\mathcal{A})$ is the Grothendieck ring of the (2-)category of Artin stacks over $\mathcal{M}_\mathcal{A}$, tensored with $\mathbb{Q}$.

Then $\mathrm{SF}(\mathcal{M}_\mathcal{A})$ has an associative, non-commutative product $*$ making it into a $\mathbb{Q}$-algebra, a kind of universal Ringel–Hall algebra. For $f, g \in \mathrm{SF}(\mathcal{M}_\mathcal{A})$, think of $(f * g)(F)$ as the 'integral' of $f(E)g(G)$ over all exact sequences $0 \to E \to F \to G \to 0$ in $\mathcal{A}$.

This $*$ induces a Lie bracket $[\,,\,]$ on $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$ by $[f,g] = f*g-g*f$. There is a vector subspace $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ of $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$, the stack functions 'supported on (virtual) indecomposables', which is closed under $[\,,\,]$ (though not under $*$). Thus $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ is a *Lie subalgebra* of $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$.

Given a stability condition τ on $\mathcal{A}$, we define elements $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau)$ for $\alpha \in K(\mathcal{A})$ to be the 'characteristic function' of $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$, regarded as a substack of $\mathcal{M}_{\mathcal{A}}$.

We prove a *universal transformation law* for the $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau)$ under change of stability condition. That is, given two stability conditions $\tilde{\tau},\tau$ on $\mathcal{A}$, we can write $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tilde{\tau})$ as a sum of products of $\bar{\delta}_{\mathrm{ss}}^{\beta}(\tau)$ using $*$, with combinatorial coefficients in $\mathbb{Z}$.

If $\mathcal{M}^\alpha_{\rm st}(\tau)\neq\mathcal{M}^\alpha_{\rm ss}(\tau)$ then $\bar\delta^\alpha_{\rm ss}(\tau)$ does not lie in the Lie subalgebra $\mathrm{SF}^{\rm ind}(\mathcal{M}_{\mathcal{A}})$. Using the $\bar\delta^\alpha_{\rm ss}(\tau)$ we define elements $\bar\epsilon^\alpha(\tau)$ for $\alpha\in K(\mathcal{A})$ which do lie in $\mathrm{SF}^{\rm ind}(\mathcal{M}_{\mathcal{A}})$. We have $\bar\epsilon^\alpha(\tau)=\bar\delta^\alpha_{\rm ss}(\tau)$ if $\mathcal{M}^\alpha_{\rm st}(\tau)=\mathcal{M}^\alpha_{\rm ss}(\tau)$. Think of $\bar\epsilon^\alpha(\tau)$ as a weighted version of $\bar\delta^\alpha_{\rm ss}(\tau)$, where stables have weight 1, indecomposable semistables have weights in $\mathbb{Q}$, and decomposables have weight 0.

The $\bar\epsilon^\alpha(\tau)$ also satisfy a *universal transformation law* under change of stability condition, with coefficients in $\mathbb{Q}$. It can be written solely using the Lie bracket $[\,,\,]$ on $\mathrm{SF}^{\rm ind}(\mathcal{M}_{\mathcal{A}})$, rather than $*$ on $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$.

All the above works for very general abelian categories, e.g. $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth projective variety over $\mathbb{K}$ algebraically closed of characteristic zero.

Now we use the Calabi–Yau 3-fold assumption. Define a Lie algebra $C(\mathcal{A},\chi)$ to have basis, as a $\mathbb{Q}$-vector space, symbols c^α for $\alpha\in K(\mathcal{A})$, and Lie bracket

$$[c^\alpha,c^\beta]=\chi(\alpha,\beta)\,c^{\alpha+\beta}, \qquad (4)$$

where χ is the Euler form. As χ is antisymmetric this satisfies the Jacobi identity. We define a linear map $\Psi:\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})\rightarrow C(\mathcal{A},\chi)$ by $\Psi(f)=\sum_{\alpha\in K(\mathcal{A})}\overline{\chi}(f|_{\mathcal{M}_{\mathcal{A}}^\alpha})c^\alpha$, where $\mathcal{M}_{\mathcal{A}}^\alpha$ is the substack of sheaves in class α in $\mathcal{M}_{\mathcal{A}}$, and $\overline{\chi}$ is a kind of stack-theoretic Euler characteristic.

Here $\overline{\chi}$ is not easy to define. The natural Euler characteristic of a quotient stack $[X/G]$ should be $\overline{\chi}([X/G])=\chi(X)/\chi(G)$, but $\chi(G)=0$ for any algebraic group of positive rank, so we have to divide by zero.

The point about $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ is that we can write $f \in \mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ using only $[X/G]$ with rank$(G) = 1$, and then set $\overline{\chi}([X/G]) = \chi(X)/\chi(G/\mathbb{C}^\times)$, where $\mathbb{C}^\times$ is the maximal torus of G, and $\chi(G/\mathbb{C}^\times) \neq 0$.

Using the Calabi-Yau 3-fold property, equation (1), we can show that $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}}) \rightarrow C(\mathcal{A}, \chi)$ is a *Lie algebra morphism*.

We then define invariants $J^\alpha(\tau) \in \mathbb{Q}$ by $\Psi(\overline{\epsilon}^\alpha(\tau)) = J^\alpha(\tau) c^\alpha$ for all $\alpha \in K(\mathcal{A})$.

Since the $\overline{\epsilon}^\alpha(\tau)$ satisfy a universal transformation law in the Lie algebra $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ under change of stability condition, and Ψ is a Lie algebra morphism, the images $J^\alpha(\tau) c^\alpha$ satisfy the same transformation law in the Lie algebra $C(\mathcal{A}, \chi)$.

This yields a transformation law for the $J^\alpha(\tau)$ under change of stability condition, of the form

$$J^\alpha(\tilde\tau) = \sum_{\substack{\text{iso. classes} \\ \text{of } \Gamma, I, \kappa}} \pm\, U(\Gamma, I, \kappa; \tau, \tilde\tau) \cdot \prod_{i \in I} J^{\kappa(i)}(\tau) \cdot \prod_{\substack{\text{edges} \\ i - j \text{ in } \Gamma}} \chi(\kappa(i), \kappa(j)). \tag{5}$$

Here Γ is a connected, simply-connected undirected graph with vertices I, $\kappa : I \to K(\mathcal{A})$ has $\sum_{i\in I} \kappa(i) = \alpha$, and $U(\Gamma, I, \kappa; \tau, \tilde\tau)$ in $\mathbb{Q}$ are explicit combinatorial coefficients.

3. A Lie algebra morphism $\tilde{\Psi} : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{C}(X)$

We can now explain our new work. We want to modify the Lie algebra morphism Ψ by inserting Behrend's microlocal function ν as a weight in its definition of Ψ, to get a new Lie algebra morphism $\tilde{\Psi}$. As ν is a 'multiplicity function', the new *generalized D–T invariants* $\bar{DT}^\alpha(\tau)$ we define using $\tilde{\Psi}$ will count sheaves with multiplicity, and so they will be unchanged under deformations of X.

Surprisingly, we also have to change the signs in the Lie algebra $C(X)$.

Define a Lie algebra $\tilde{C}(X)$ to have basis, as a $\mathbb{Q}$-vector space, symbols $\tilde{c}^\alpha$ for $\alpha \in K(\mathcal{A})$, and Lie bracket

$$[\tilde{c}^\alpha, \tilde{c}^\beta] = (-1)^{\chi(\alpha,\beta)} \chi(\alpha, \beta)\, \tilde{c}^{\alpha+\beta}, \qquad (6)$$

which is (4) with an extra factor $(-1)^{\chi(\alpha,\beta)}$.

Define a linear map $\tilde{\Psi}:\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})\rightarrow\tilde{C}(X)$ by $\tilde{\Psi}(f)=\sum_{\alpha\in K(\mathcal{A})}\overline{\chi}(f|_{\mathcal{M}^\alpha_{\mathcal{A}}},\nu)\tilde{c}^\alpha$, where $\overline{\chi}(\cdots,\nu)$ is $\overline{\chi}$ weighted by ν.

Theorem. $\tilde{\Psi}:\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})\rightarrow\tilde{C}(X)$ *is a Lie algebra morphism*.

This follows from my previous proof that Ψ is a Lie algebra morphism, together with two multiplicative identities for the Behrend function ν, that is

$$\nu(E_1\oplus E_2)=(-1)^{\chi([E_1],[E_2])}\nu(E_1)\nu(E_2), \quad (7)$$

$$\int_{\epsilon\in P(\mathrm{Ext}^1(E_2,E_1))}\nu(F)\mathrm{d}\chi-\int_{\epsilon\in P(\mathrm{Ext}^1(E_1,E_2))}\nu(F)\mathrm{d}\chi= \quad (8)$$

$$\Bigl(\dim\mathrm{Ext}^1(E_2,E_1)-\dim\mathrm{Ext}^1(E_1,E_2)\Bigr)\nu(E_1\oplus E_2),$$

where in the first integral in (8), F is defined in terms of ϵ such that the exact sequence $0\rightarrow E_1\rightarrow F\rightarrow E_2\rightarrow 0$ corresponds to $\epsilon\in P(\mathrm{Ext}^1(E_2,E_1))$, and similarly for the second integral.

4. Proving the Behrend function identities (7),(8)

Let $\mathfrak{F}$ be a $\mathbb{C}$-scheme or Artin $\mathbb{C}$-stack, locally of finite type. The Behrend function $\nu_{\mathfrak{F}}$ is a $\mathbb{Z}$-valued constructible function on $\mathfrak{F}$ which measures the 'multiplicity' of $\mathfrak{F}$ at each point. In general it is difficult to compute. But there is a special case in which we can give an explicit formula for $\nu_{\mathfrak{F}}$: suppose $\mathfrak{F}$ is a $\mathbb{C}$-scheme, U is a complex manifold, $f : U \rightarrow \mathbb{C}$ is holomorphic, and $\mathfrak{F}$ is locally isomorphic (in the analytic topology) to $\mathrm{Crit}(f)$ as a complex analytic space. Then

$$\nu_{\mathfrak{F}}(x) = (-1)^{\dim U}\bigl(1 - \chi(MF_f(x))\bigr),$$

with $MF_f(x)$ the *Milnor fibre* of f at x.

Our proof of (7),(8) involves first showing that we can write an atlas for the moduli stack $\mathfrak{M}$ of coherent sheaves on a Calabi–Yau 3-fold X over $\mathbb{C}$ in the form Crit(f) locally in the analytic topology, for f a holomorphic function on a complex manifold U. Note that f,U are *not* algebraic, they are constructed by transcendental, gauge-theoretic methods. Our proof works only over $\mathbb{C}$, not for more general fields $\mathbb{K}$.

The proof has three steps:

(a) Show that the moduli stack $\mathfrak{M}$ of coherent sheaves on X is locally isomorphic (in the Zariski topology) to the moduli stack $\mathfrak{Vect}$ of vector bundles on X. (This works for Calabi–Yau m-folds X over $\mathbb{K}$ for any $m, \mathbb{K}$.)

(b) Show that an atlas for $\mathfrak{Vect}$ near $[E]$ can be locally written in the form Crit(f) for $f : U \to \mathbb{C}$, where f, U are invariant under at least the maximal compact subgroup of Aut(E).

(c) Prove (7),(8) using an atlas near $E = E_1 \oplus E_2$, and localizing under the action of the $U(1)$ group $\{\mathrm{id}_{E_1} + \lambda\, \mathrm{id}_{E_2} : \lambda \in U(1)\}$.

For (a), one uses Seidel–Thomas twists to show the local equivalence of moduli of sheaves and vector bundles. Given an integer n, the Seidel–Thomas twist with $\mathcal{O}_X(-n)$, T_n, is the Fourier-Mukai transform from $D(X)$ to $D(X)$ with kernel:

$$\mathrm{cone}(\mathcal{O}_X(n) \boxtimes \mathcal{O}_X(-n) \to \mathcal{O}_\Delta).$$

Given $E \in \mathfrak{M}$. For $n \gg 0$, $F = T_n(E)$ is a *sheaf*, not a more general complex, and we have:

$$0 \to F \to \mathcal{O}_X(-n) \otimes H^0(E(n)) \to E \to 0.$$

As X is Calabi–Yau, T_n induces local isomorphisms of moduli spaces. We inductively define integers $n_1, \ldots, n_m \gg 0$ and set $F_i = T_{n_i} \circ T_{n_{i-1}} \ldots \circ T_{n_1}(E)$. We get an exact sequence:

$$\begin{aligned} 0 \to \quad & F_m \to \mathcal{O}_X(-n_m) \otimes H^0(F_{m-1}(m)) \to \\ \ldots \to \quad & \mathcal{O}_X(-n_1) \otimes H^0(E(n_1)) \to E \to 0. \end{aligned}$$

By the Hilbert Syzygy Theorem, F_m is a vector bundle.

For (b), we use an idea of Richard Thomas. Let $E \to X$ be a fixed complex (not holomorphic) vector bundle. The holomorphic structures on E are $\bar\partial$-operators $\bar\partial_E : C^\infty(E) \to C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}T^*X)$. The set of such $\bar\partial$-operators is an infinite-dimensional affine space $\mathcal{A}$. A $\bar\partial$-operator $\bar\partial_E$ is a holomorphic structure iff the $(0,2)$-curvature $\bar\partial_E^2$ is zero. Gauge transformations $\mathcal{G} = C^\infty(\mathrm{Aut}(E))$ act on $\mathcal{A}$. Thus, the moduli space (stack) of holomorphic structures on E up to isomorphisms is

$$\mathcal{M}_E = \{\bar\partial_E \in \mathcal{A} : \bar\partial_E^2 = 0\}/\mathcal{G}.$$

Richard observed that $\{\bar\partial_E \in \mathcal{A} : \bar\partial_E^2 = 0\}$ is $\mathrm{Crit}(CS)$, in some infinite-dimensional manifold sense, where $CS : \mathcal{A} \to \mathbb{C}$ is the *holomorphic Chern–Simons functional*.

To prove (b), we show that an atlas for $\mathfrak{Vect}$ near $(E, \bar{\partial}_E)$ can be written locally as Crit$(CS|_U)$, where U is a finite-dimensional complex submanifold of $\mathcal{A}$, which is roughly speaking transverse to the orbit of $\mathcal{G}$ through $\bar{\partial}_E$. We use results of Miyajima and others which locally identify the moduli spaces of holomorphic structures on E, and of analytic vector bundles on X, and of algebraic vector bundles on X.

To prove (c): let $E = E_1 \oplus E_2$ be a coherent sheaf on X. Then (a),(b) show that we can write an atlas for $\mathfrak{M}$ near E as Crit(f) near 0, where f is a holomorphic function defined near 0 on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$, and f is invariant under the action of $T = \{\mathrm{id}_{E_1} + \lambda\,\mathrm{id}_{E_2} : \lambda \in U(1)\}$ on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ by conjugation.

The fixed points of T on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ are $\mathrm{Ext}^1(E_1, E_1) \oplus \mathrm{Ext}^1(E_2, E_2)$, and that the restriction of f to these fixed points is $f_1 + f_2$, where f_j is defined near 0 in $\mathrm{Ext}^1(E_j, E_j)$, and Crit(f_j) is an atlas for $\mathfrak{M}$ near E_j.

The Milnor fibre $MF_f(0)$ is invariant under T, so by localization we have

$$\chi(MF_f(0)) = \chi(MF_f(0)^T) = \chi(MF_{f_1+f_2}(0)).$$

The Thom–Sebastiani theorem gives

$$1 - \chi(MF_{f_1+f_2}(0)) = (1 - \chi(MF_{f_1}(0)) (1 - \chi(MF_{f_2}(0)).$$

Equation (7) then follows easily from

$$\nu_{\mathfrak{M}}(E) = (-1)^{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Hom}(E,E)} (1 - \chi(MF_f(0)),$$

and the analogues for E_1, E_2. Equation (8) uses a more involved argument to do with Milnor fibres of f at non-fixed points of the $U(1)$-action.

5. Generalized D–T invariants

We then define invariants $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ by $\tilde{\Psi}(\bar{\epsilon}^\alpha(\tau)) = \bar{DT}^\alpha(\tau)\tilde{c}^\alpha$ for all $\alpha \in K(\mathcal{A})$. Since $\tilde{\Psi}$ is a Lie algebra morphism, and the $\bar{\epsilon}^\alpha(\tau)$ satisfy a universal transformation law under change of stability condition, it follows that the $\bar{DT}^\alpha(\tau)$ satisfy a known transformation law under change of stability condition. When $\mathcal{M}_{\mathrm{st}}^\alpha(\tau) = \mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$ we have $\bar{\epsilon}^\alpha(\tau) = \bar{\delta}_{\mathrm{ss}}^\alpha(\tau)$, giving

$$\bar{DT}^\alpha(\tau) = \int_{\mathcal{M}_{\mathrm{st}}^\alpha(\tau)} \nu \,\mathrm{d}\chi = \mathrm{DT}^\alpha(\tau) \quad (9)$$

by (2). Thus, the $\bar{DT}^\alpha(\tau)$ are generalizations of Donaldson–Thomas invariants.

It remains to show that the $\bar{DT}^\alpha(\tau)$ are unchanged under deformations of the underlying Calabi–Yau 3-fold X.

To do this we define an auxiliary invariant $PI^\alpha(N,\tau)\in\mathbb{Z}$ counting 'stable pairs' (E,s) with E a semistable sheaf in class α and $s\in H^0(E\otimes\mathcal{L}^N)$, for $N\gg 0$, where $\mathcal{L}$ is the ample line bundle used to define τ. By a similar proof to Pandharipande–Thomas invariants, the moduli space of stable pairs has a symmetric obstruction theory, so $PI^\alpha(N,\tau)$ is unchanged by deformations of X.

We then prove that $PI^\alpha(N,\tau)$ can be written in terms of the $\bar{DT}^\beta(\tau)$ by

$$PI^\alpha(N,\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in K(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha)\ \forall i}}\frac{(-1)^n}{n!}\prod_{i=1}^{n}\begin{array}{l}(-1)^{\chi([\mathcal{L}^{-N}]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)}\cdot\\ \chi([\mathcal{L}^{-N}]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)\bar{DT}^{\alpha_i}(\tau).\end{array}\tag{10}$$

We prove (10) using a change of stability condition formula in an auxiliary abelian category $\mathcal{B}$, whose objects are triples (V, E, ϕ) for V a finite-dimensional $\mathbb{C}$-vector space, E a coherent sheaf, and $\phi : V \to H^0(E \otimes \mathcal{L}^N)$ a linear map. Now (10) implies that

$$PI^\alpha(N,\tau) = (-1)^{\chi([\mathcal{L}^{-N}],\alpha)}\chi([\mathcal{L}^{-N}],\alpha)\bar{DT}^\alpha(\tau) + \cdots,$$

where the lower order terms '$\cdots$' involve only $\bar{DT}^\beta(\tau)$ with $\dim\beta = \dim\alpha$ and $\operatorname{rank}\beta < \operatorname{rank}\alpha$.
Also $\chi([\mathcal{L}^{-N}],\alpha) = \dim H^0(E \otimes \mathcal{L}^N) > 0$ for $N \gg 0$. Thus, fixing $\dim\alpha$ and arguing by induction on $\operatorname{rank}\alpha$, since $PI^\alpha(N,\tau)$ is deformation-invariant, we see that $\bar{DT}^\alpha(\tau)$ is deformation-invariant.

Integrality properties of the invariants
Suppose E is stable and rigid in class α. Then $kE = E\oplus\cdots\oplus E$ is strictly semistable in class $k\alpha$, for $k \geqslant 2$. Calculations show that E contributes 1 to $\bar{DT}^{\alpha}(\tau)$, and kE contributes $1/k^2$ to $\bar{DT}^{k\alpha}(\tau)$. So we do not expect the $\bar{DT}^{\alpha}(\tau)$ to be integers, in general.

Define new invariants $KS^{\alpha}(\tau) \in \mathbb{Q}$ by

$$\bar{DT}^{\alpha}(\tau) = \sum_{k\geqslant 1:k \text{ divides } \alpha} \frac{1}{k^2} KS^{\alpha/k}(\tau).$$

Then the kE for $k \geqslant 1$ above contribute 1 to $KS^{\alpha}(\tau)$ and 0 to $KS^{k\alpha}(\tau)$ for $k > 1$.

Conjecture. *Suppose τ is generic, in the sense that $\tau(\alpha) = \tau(\beta)$ implies $\chi(\alpha,\beta) = 0$. Then $KS^{\alpha}(\tau) \in \mathbb{Z}$ for all $\alpha \in K(\mathcal{A})$.*

These $KS^{\alpha}(\tau)$ may coincide with invariants conjectured by Kontsevich–Soibelman, and in String Theory should perhaps be interpreted as 'numbers of BPS states'.