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K-equivalence in GW theory

Yuan-Pin Lee
2009/2/25

* (Most of) the talk is based on $\boxed{\text{joint work w/ C-L Wang \& H-W Lin.}}$ (National Taiwan University)

Def'n. 2 (sm / $\mathbb{Q}$-Gorenstein) varieties X & X' are K-equivalent if $\exists$ birational morphism $\phi : Y \to X$ & $\phi' : Y \to X'$ s.t. $\phi^* K_X = \phi'^* K_{X'}$.

crepant birational transf.

Main example of K-equivalence : $\boxed{\text{Flops}}$

Def: ① Let $\psi : X \to \bar{X}$ be a flopping contraction
$$\begin{matrix} X & \to & \bar{X} \\ \cup & & \cup \\ \bar{\psi} : Z & \to & S \end{matrix}$$
exceptional

Assume: $\cdot\ Z = \mathbb{P}_S(F)$ $\quad F \to S$ v.b. of rk $= r$

$\cdot\ N_{Z|X}\big|_s = \mathcal{O}_{\mathbb{P}^r}(-1)^{r+1}$ $\quad \forall s \in S$.

The corresp. flop $X \dashrightarrow X'$ is called an $\boxed{\text{ordinary flop}}$

② ~~ψ~~ is called $\boxed{\text{Mukai flop}}$ if $N_{Z|X} = T^*_{Z/S}$ (rel cotang)

Thm: $\cdot$ GW(X) ~~$\cong$~~ $\sim$ GW(X')
for Mukai flops & ordinary flops (work in progress) at most

② ~~Genus (K ≥ 0)~~ $g \geq 0$ ~~(K ≥ 0)~~

Rmk: $\cdot$ Since GW theory is inv under sympl. deformation,
$\to$ e.g. 3-dim. & "predictive"
$\cdot$ $\sim$ is ~~of predic~~ "exact" in the sense that
Knowing GW(X) gives GW(X') + classical & vice versa

$\cdot$ Non-toric !

$\cdot$ No ad hoc comparison

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<u>Def'n</u> [Flopping contraction]

$\phi : X \to \overline{X}$ is
a small contraction of an extremal ray ℓ w.r.t. $K_X + \epsilon D$
$0 < \epsilon \ll 1$, $D > 0$ s.t. $K_X \cdot \ell = 0$ & $-D$ is ϕ-ample

(In other words, ℓ is <u>log-extremal</u> w.r.t. $(K_X + \epsilon D)$.)

$$\Updownarrow$$

$$\begin{cases} (K_X + \epsilon D) \cdot \ell < 0 \\ C \text{ is extremal} \end{cases}$$

<u>e.g.</u> <u>Atiyah flop</u>

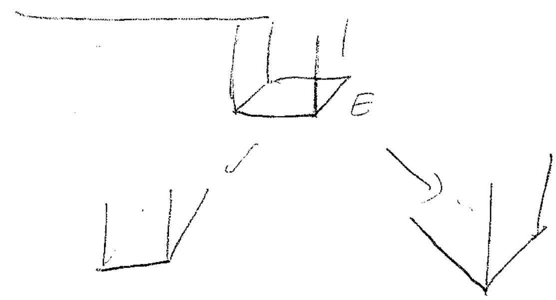
$$\overline{X} = \{ z_1 z_4 - z_2 z_3 = 0 \} \subset \mathbb{A}^4$$

$$\mathbb{P}^1 \times \mathbb{P}^1 = E \subset Y = Bl_0 \overline{X} \qquad N_{E|Y} = \mathcal{O}(-1,-1)$$

$$\begin{tikzcd}
 & E \arrow[dl, "\pi_1"'] \arrow[dr] & Y \arrow[dl] \arrow[dr] & \\
\mathcal{O}(-1)\oplus\mathcal{O}(-1) \curvearrowright Z = \mathbb{P}^1 \subset X \arrow[dr] & & \mathbb{P}^1 \arrow[dl] & X' \arrow[dl, "\phi'"] \\
 & 0 \in \overline{X} \arrow[from=ul, phantom] & \overline{X} \arrow[from=ul, phantom] &
\end{tikzcd}$$

$$X \xrightarrow{\phi} \overline{X} \xleftarrow{\phi'} X'$$

$$D_\phi = \mathbb{P}^1_{(2)} \qquad D_{\phi'} = \mathbb{P}^1_{(1)},$$



Motivations: ① Thm [Batyrev, C-L Wang, Takehiko Yasuda ...]

3/7 K-equiv var / orbifolds have the same Betti #'s

Q: ① Canonical corresp

② ring structure.

Conj [Y. Ruan, C-L Wang] K-equiv $\Rightarrow$ iso QH^* ring (500)

② Functoriality

③ Crepant resol'n

The proofs of 2 flops are completely different

· $\boxed{\text{Mukai}}$ = a "slice" of ordinary flop

· $(F, F' = F^*)$ $rk = r+1$

$\boxed{\text{local model}}$

$$\begin{tikzcd} E = \mathbb{P}_S(F) \times_S \mathbb{P}_S(F^*) \subset \mathcal{Y} \arrow[d] \arrow[dr] \arrow[drr] & & \\ Z = \mathbb{P}_S(F) \subset \mathcal{X} \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & S \subset \bar{\mathcal{X}} & \end{tikzcd}$$

$$\mathcal{O}_E(-1,-1) \to \mathcal{Y}^*(F \otimes F^*) \to \mathcal{O}_E \xrightarrow{\text{tr}} \mathbb{C}$$

$$\begin{matrix} \| & & & \cong \\ \mathcal{Y} & \xrightarrow{\pi} & & E \times \mathbb{C} \to \mathbb{C} \end{matrix}$$

Claim

Prop. $\pi^{-1}(t)$ gives an iso of $\mathcal{X}_t$ & $\mathcal{X}'_t$

$\pi^{-1}(0)$ — Mukai flop.

Thm 4.17 For any Mukai flop: $f: X \dashrightarrow X'$

- X diffeom. to X'
- iso (identical) Chow motives, Hodge structures, full GW theory.

<u>Identification</u>: $\Gamma \subset X \times X'$ <u>fiber product</u>

$$\begin{tikzcd} & X \times X' \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "\phi"'] & & X' \end{tikzcd}$$

<u>pf</u>:
- ~~Localiza~~
- <u>Degeneration formula</u> $\longrightarrow$ reduces things to projective completion of the open local models.
- extremal curve $\ell \subset X \longrightarrow -\ell' \subset X'$

  $\Rightarrow$ extremal contrib vanishes

<u>Rmk</u>: <u>Goal</u>: To compare, Don't calculate $\boxed{\text{unless necessary}}$

"Mukai flops" successful.

<u>GW theory</u>: <u>GWI</u> of X

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev_1^*(\alpha_1) \cdots ev_n^*(\alpha_n)$$

β is an effective curve class in $N_1(X)$

Ordinary flops.

5/7. $\mathcal{F}: H^*(X) \dashrightarrow H^*(X')$ graph closure = fiber product

$\ell \xmapsto{\mathcal{F}} -\ell'$

Thm. $\mathcal{F}$ induces equivalence of Chow motives

In particular, it preserves the "Poincaré pairing"

$(\alpha_1, \alpha_2)^X := (\mathcal{F}(\alpha_1), \mathcal{F}(\alpha_2))^{X'}$ $\deg(\alpha_1) + \deg(\alpha_2) = \dim X$

- However: the triple products ($\Leftrightarrow$ i.e. ring structures) are NOT preserved

e.g. $\int_{X'} \mathcal{F}(\alpha_1) \cdot \mathcal{F}(\alpha_2) \cdot \mathcal{F}(\alpha_3) = \int_X \alpha_1 \alpha_2 \alpha_3 + (-1)^r (\alpha_1 \cdot h^{r-l_1})(\alpha_2 \cdot h^{r-l_2})(\alpha_3 \cdot h^{r-l_3})$

Thm. • For simple flops: The full GW theory (ancestors) is identified by $\mathcal{F}$. all genera

[In particular $\ell \beta$ in $N_1(X) \to -\ell' \in N_1(X')$

$\Rightarrow q^{\ell} \mapsto \frac{1}{q^{\ell'}}$ ← $\mathbb{P}^1_1$-family (a pt) of GWT]

• For ordinary flops in $g=0$

(Short of the last step.)

$g=0$ $GW(X)$ $q=\infty$ $GW(X')$

$\boxed{\text{Clean statement of results so far}}$ "$\beta = d\ell$"

$g=0$ GW theory (ancestor) ass. w/ the extremal ray is ~~the one~~ identified by $\mathcal{F}$.

Conj: Analyticity ~~for~~ of ~~l~~ extremal ray for flopping contraction

How to prove the statement ?

6/7 . Note: X & X' are birational

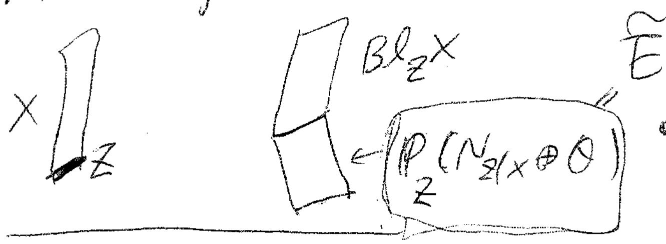
$\Rightarrow$ iso except on $\underbrace{Z \quad Z'}_{\text{excep. loci}}$

$\Rightarrow$ Want to "localize" (Not in the sense of Borel / gp action) to the neighborhood of Z & Z'

$\rightarrow$ Unfortunately, there is NO "motivic" descript. NOT in GWT (can give sheaf)

However, $\exists$ degeneration formula (A. Li - Y. Ruan, JL (diff. formulat by Ionel-Parker)

$X \times \mathbb{A}^1$ deg to normal cone



"projective completion of "tubular neighborhd"

Degen. formula $\rightsquigarrow$ problem reduced to "proj local model".

"proj local model": ~~is~~ a double projective bundle over S

$$\begin{array}{l} \tilde{E} = \mathbb{P}_Z(N_{Z/\tilde{E}} \oplus \mathcal{O}) \\ \downarrow \\ Z = \mathbb{P}_S(F) \\ \downarrow \\ S \end{array} \quad \pi$$

$S = \underline{pt}$ "simple flop" $\leftarrow$ standard flop

$\tilde{E} = \underline{\text{toric}}$ $\leftarrow$ ~~top~~ GWI "in principle" computable

nevertheless, ~~com~~ $\boxed{\text{MOTTO}}$:

$\boxed{\text{Compare, don't compute}}$.

esp. one needs analytic continuation.

Talk #9 1/7

K-equivalence in Gromov-Witten Theory

Yuan-Pin Lee
Feb 25, 2009
11 AM - 12 PM

Joint work with H-W Lin + C-L Wang
(National Taiwan University)

Def Two sm varieties (......) X + X' are <u>K-equivalent</u> if $\exists$ birat'l morphism s.t. $\phi^* K_X = \phi'^* K_{X'}$.

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ X & & X' \end{tikzcd}$$

Y smooth.

Also known as crepant transformation, etc.

<u>Main examples</u> for today, two kinds of flops

(i) <u>ordinary flops</u>

$$\begin{tikzcd} E \subseteq Y \arrow[r] & Z \subseteq X \arrow[r, dashed, "\text{birat'l}"] \arrow[d, "\text{fl. contraction}"] \arrow[dr, phantom, "\psi"] & X' \arrow[dl] \\ & \bar{X} & \\ & S \arrow[u, "\iota"] & \end{tikzcd}$$

exceptional locus

$Y = Bl_Z X$, $\bar{\psi}: Z \to S$

- $\exists$ v.b. $F \to S$ s.t. $Z = \mathbb{P}_S(F)$ (rk $= r+1$)
- $\forall s \in S$ $Z_s \cong \mathbb{P}^r$, and $N_{Z/X}\big|_{s\in S} = \mathcal{O}(-1)^{r+1}$

(ii.) <u>Mukai flops</u> same as above setting, except we require •• $N_{Z/X} \cong T^*_{\bar{\psi}}$.

normal bundle of Z inside X is isom to the relative tangent bundle of $\bar{\psi}$.

for all genera

Goal: To show $GW(X) \xrightarrow{\sim} GW(X')$ in these two cases.

Thm
- Goal ✓ for Mukai flops
- Goal ✓ for simple flops ($S = pt$)
- Goal partially done for genus 0 and F, F' split, $F = \bigoplus_i L_i$, $F' = \bigoplus_i L_i'$

Rmk
- G-W theory is invariant under sympl. deform.
  $\Rightarrow$ In dim $= 3$, only have to consider Atiyah flops ($r = 1$, $S = pt$).
  Easy to group sheaves, but not GW invariants.
- Realize the goal in an "exact" and "predictive" way in the sense that if $GWT(X)$ is known, $\Rightarrow$ complete knowledge of $GW(X')$ w/o calculating anything on X' side.

Motivations

(i) Thm [Batyrev, Wang, Yasuda] K-equivalent varieties have the same Betti #'s.

$$H^*(X)_{\mathbb{Q}} \qquad H^*(X')_{\mathbb{Q}}$$

Q1. Is there a canonical correspondence between $H^*(X)$ and $H^*(X')$?

<u>Q 2</u>. Suppose Yes for Q1.
Then does that preserve the ring structures?

Answer for Q1: Yes for ordinary and Mukai flops.

$$\overset{\text{cycle}}{\Gamma} \subseteq X \times X'$$

$$\begin{tikzcd} & X \times X' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr] & & X' \arrow[dl] \\ & \bar{X} & \end{tikzcd}$$

$\mathcal{F} : H^*(X) \to H^*(X')$
by
$\pi_{2*}(\pi_1^*(\alpha) \cdot \Gamma) =: \mathcal{F}(\alpha)$

Γ for ordinary flops = graph closure.
Γ for Mukai flops = fiber product =
= graph closure $\cup$ excep. comp.

<u>Q1</u> is answered affirmatively.

$\mathcal{F}$ preserves "Chow motives"!
$\mathcal{F}$ preserves Poincaré pairing.

<u>Q2</u>. $\mathcal{F}$ is NOT a ring homom.

In the case of Atiyah flops,

$$\boxed{\begin{array}{c}\sum \deg(\alpha_i) \\ = \dim(X)\end{array}}$$

$$\int_{X'} \mathcal{F}(\alpha_1)\mathcal{F}(\alpha_2)\mathcal{F}(\alpha_3) = \int_X \alpha_1 \alpha_2 \alpha_3 + \underline{\text{nonzero error term.}}$$

$$(-1)^r(\alpha_1 h^{r-l_1})(\alpha_2 h^{r-l_2})(\alpha_3 h^{r-l_3})$$

h = hyperplane in $\mathbb{P}^r = S$

But if one takes instead the quantum ring structure, then $\mathcal{F} : QH^*(X) \xrightarrow{\cong} QH^*(X')$.

To prove a statement like this,

Motto: To compare, NOT to compute unless necessary.

<u>For Mukai flops</u>
Realize Mukai flops (local picture) as a slice of ordinary flops.

For $t=0$, get Mukai flops

$F' = F^*$

$$\begin{tikzcd}
\mathbb{P}_Z(N_{Z|X}) = E \subseteq Y \arrow[rr, "t"] \arrow[d, "\text{blow up}"'] \arrow[drr, "\text{then blow down}"] & & \mathbb{C} \\
\mathbb{P}_S(F) = Z \subseteq X \arrow[dr] & & \mathbb{P}_S(F^*) = Z' \subseteq X' \arrow[dl] \\
& S \subseteq \bar{X} &
\end{tikzcd}$$

S: exceptional locus

$N_{E|Y} = \mathcal{O}(-1,-1)$ an open local model for Y

$$\mathcal{O}(-1,-1) \to F \otimes F^* \to \mathcal{O}_E \cong E \times \mathbb{C} \to \mathbb{C} \ni t$$

Claim: • $\pi^{-1}(0)$ is a picture of Mukai flops X_0 & X_0'.
• $\pi^{-1}(t)$ give an isom. between X_t and X_t'

Consequence. locally, Mukai flops are "limits" of isom.

$\ell \subseteq X \quad \ell' \subseteq X'$
$\ell \xmapsto{\mathcal{F}} -\ell'$

$$\Rightarrow GW(X) \xrightarrow[\cong]{\mathcal{F}} GW(X')$$

(No analytic continuation needed.)

For ordinary flops

GWI: $$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} ev_1^*(\alpha_1) \ldots ev_n^*(\alpha_n)$$

β: curve class — global ble curve move around — lies in Mori cone

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n} = \sum_{\beta \in NE(X)} q^\beta \langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta}$$

Formal power series

$$\mathcal{F} : GW(X) \xrightarrow{\cong} GW(X')$$

$$\beta = \ell \longmapsto \mathcal{F}(\beta) = -\ell'$$

$$q^\ell \longmapsto q^{-\ell'} \quad (\text{problem!})$$

not well-defined in GWT.

Guess. ℓ = extremal ray of a flopping contraction

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{ext}_{g,n}(q^\ell) = \sum_{d=1}^{\infty} \langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,d\ell}\, q^{d\ell}$$

is an analytic fn. on q^ℓ
(in fact, is a rat'l function on $\mathbb{P}^1 \setminus \{pt\}$.)

Prop. Guess holds for ordinary flops.

Goal': $GW(X)_{q^{\ell} = q}$ $\quad$ $GW(X')_{q^{\ell'} = \frac{1}{q}}$

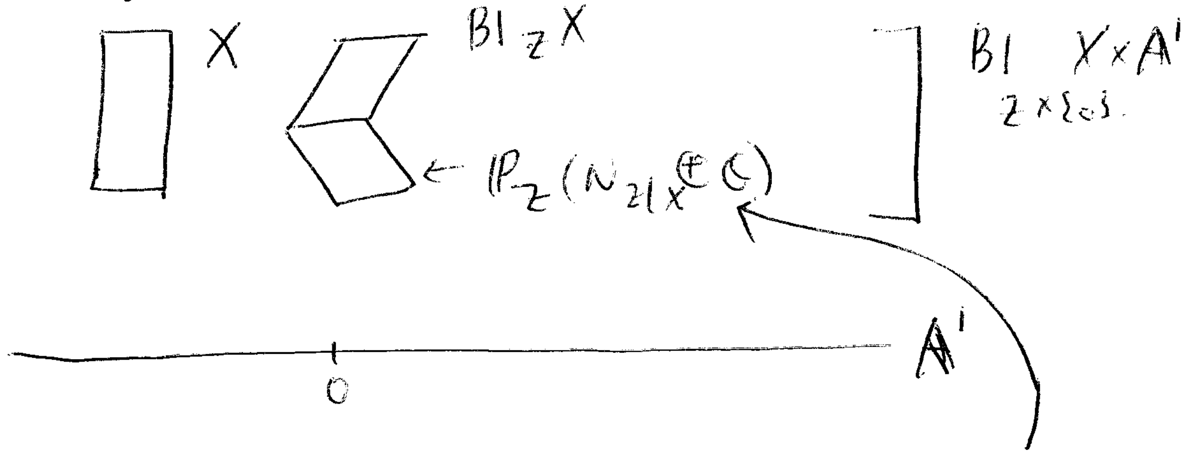
come from a family of GWT's $\to q \in \mathbb{P}^1 \setminus \{pt\}$

$GW(X)$ = formal power series exp. at $q = 0$
$GW(X')$ = " " " " " $q = \infty$

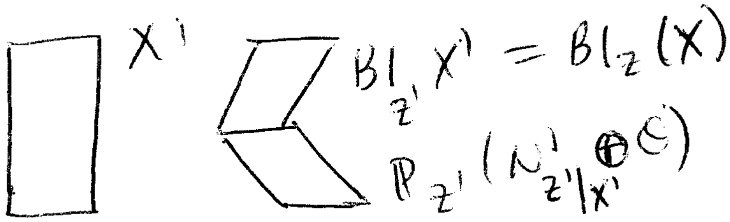
To achieve this goal

"Localize" to exceptional loci.
Use degeneration formula [Li-Ruan, J. Li]
[Ionel-Parker]

Degeneration to normal cone



proj. completion of open local picture.



7/7 $Z = \mathbb{P}_S(F)$

"<u>Goal</u>" Compare $GW(\mathbb{P}_Z(N \oplus \mathcal{O}))$
with $GW(\mathbb{P}_{Z'}(N' \oplus \mathcal{O}))$

When $S = pt$, all local picture becomes toric. In principle, computable, and completely realized.

When $S \neq pt$, what we have so far is if

- $F = \oplus L_i$, $F' = \oplus L_i'$
- $\langle \alpha_1, \ldots, \alpha_n \rangle^X_{0,n}(q) = \sum_d q^{d\ell} \langle \alpha_1, \ldots, \alpha_n \rangle$

ext. ray

$$\langle \mathcal{F}(\alpha_1), \ldots, \mathcal{F}(\alpha_n) \rangle^{X'}_{0,n}(q) =$$

$$= \langle \alpha_1, \ldots, \alpha_n \rangle^X_{0,n}\left(\frac{1}{q}\right)$$

for $n \geq 3$.