

The Verlinde bundles in higher genus

Two apologies: the moduli spaces studied here are rather classical; the word "modern" may not apply. The second is more serious: there seems to be two different Verlinde bundles in the literature; they live on different moduli spaces and have different flavors. I will only speak about one of them, to the disappointment of the audience.

[1] Setup **[a]** let $X =$ smooth complex projective curve of genus $g \geq 1$. Classical objects associated to X are:

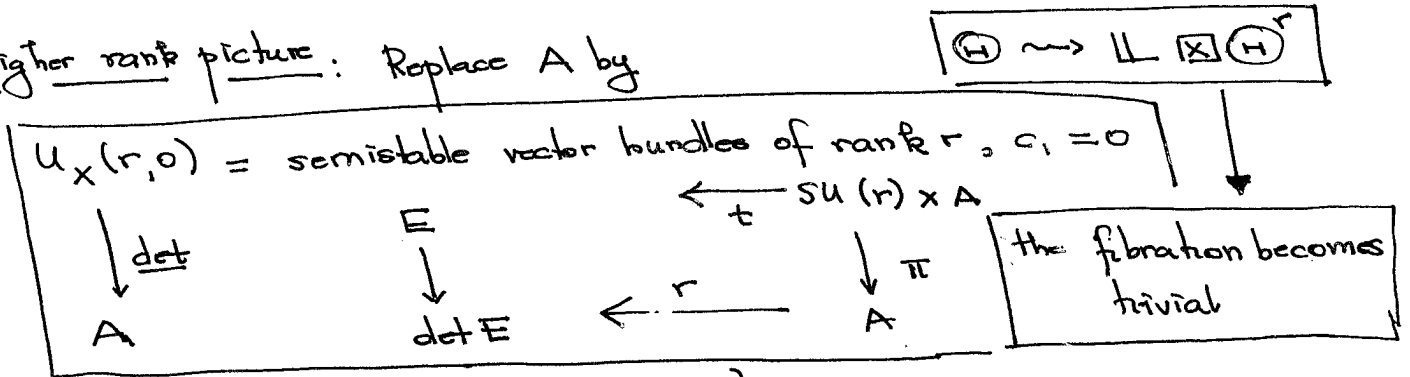
- the jacobian $A = \text{Jac}^0(X) \ni [L] =$ degree 0 line bundle. (abelian variety)

- principal polarization: let $K^2 = \omega_X$ be a theta characteristic. Define

$\Theta_K = \{L : h^0(L \otimes K) \neq 0\}$. is a divisor in A : Θ is symmetric $(-1)^* \Theta \cong \Theta$

- theta functions of level k : $H^0(A, \Theta^k) =$ dimension is k^g .

[b] Higher rank picture: Replace A by



Theta divisors $\Theta_K = \{E : h^0(E \otimes K) \neq 0\}$ and accordingly, theta functions of rank r and level k .

Variant of the construction: the fiber of $\det^{-1}(0) : SU_X(r, 0)$. All

Θ_K^1 's restrict to the same line bundle $\mathbb{L} \rightarrow SU_X(r, 0)$ which generates Picard.

$h^0(SU_X, \mathbb{L}^k) =$ rank r , level k , fixed trivial determinant theta's.

□ Dimension: $h^0(SU_x(r), \mathbb{L}^k) = \binom{r}{r+k}^g V_g(r, k)$ where 2/6

$$V_g(r, k) = (r+k)^{r(g-1)} \sum_{\substack{S \subset [r+k] \\ \#S=r}} \prod_{s \neq t} \left| 2 \sin \frac{s-t}{r+k} \right|^{1-g}$$

Note: level-rank symmetry: $V_g(r, k) = V_g(k, r)$.

For instance $V_g(r, 1) = V_g(1, r) = (r+1)^g \Rightarrow h^0(SU_x(r), \mathbb{L}) = r^g =$
 $= \# \text{ level } \geq \text{ theta functions. In fact:}$

□ Strange duality:

Thm (Marian-Oprea, Belkale) There is an isomorphism:

$H^0(U_x(k), \mathbb{H}^r)$ and $H^0(SU_x, \mathbb{L}^k)^\vee$ induced geometrically
 by the equation of the divisor $\{(E, F) \mid h^0(EFk) \neq 0\}$.

□ Verlinde bundles (Popa): $\mathbb{E}_{r, k} = \underline{\det}_* (\mathbb{H}^{\otimes k}) =$ bundles of
 generalized theta functions, fixed rank / level / determinant but vary determinant in \mathbb{A}^1

Corollary the Fourier-Mukai transform with respect to the Poincaré
 bundle \mathcal{P} on $\mathbb{A}^1 \times \mathbb{A}^1$ induces an isomorphism $(\mathcal{P}|_{\mathbb{A}^1 \times 0}, \mathcal{P}|_{0 \times \mathbb{A}^1} = \text{trivial})$

$$\widehat{\mathbb{E}}_{k, r} \cong \mathbb{E}_{r, k}^\vee \quad \text{where } \wedge : \alpha \rightarrow R_{\mathbb{P}^1}(g^* \alpha \cdot \mathcal{P})$$

Properties: □ $\mathbb{E}_{r, k}$ is ample, polystable, globally generated if $k \geq r+1$, Π_0
 semihomogeneous.

Over the origin: $\widehat{\mathbb{E}}_{k, r}|_0 = h^0(\mathbb{E}_{k, r}) = h^0(U_x(k), \mathbb{H}^r) = h^0(SU_x(r), \mathbb{L}^k)^\vee = \mathbb{E}_{r, k}|_0$

Today, describe Verlinde bundles in full generality.

Verlinde bundles have been used by Poppo to study:

- [i] positivity properties of the theta bundles on the moduli space $U_X(r)$.
- [ii] restriction to the curve can be used to give basepoints for $|\mathcal{L}|$ on SU_X .

-today we will describe the Verlinde bundles in full generality.

[2] Vector bundles on abelian varieties

[a] Let (A, Θ) be an abelian variety of dimension g ; $(-1)^* \Theta = \Theta$ a symmetric polarization. Moduli spaces of vector bundles over A have been studied by Mukai, and a class of vector bundles emerged as particularly important.

Def W is semihomogeneous if $t_x^* W \cong W \otimes y$ for $x \in A, y \in \hat{A}$. (Mukai/Umemura)

Examples

- [i] line bundles
- [ii] up to isogenies, W splits as sums of line bundles.
- [iii] W moves in a g -diml moduli space, isogenous to A .
- [iv] $ch W = rk W \cdot \exp(c_1(W)/rk W)$. for slope $\mu = \frac{b(W)}{a}$

Proposition For odd coprime integers, (a,b) there is a unique simple symmetric semihomogeneous $W_{a,b}$ with

$$\boxed{\text{rank } W_{a,b} = a^g \text{ and } \det W_{a,b} = a^{g-1} b \Theta}$$

Moreover, [i] $a^* W_{a,b} \cong \bigoplus \Theta^{ab} \cong \Theta^{ab} \otimes a^g$ -diml vector space

[ii] $W_{a,b} \otimes (a\text{-torsion}) \cong W_{a,b}$. Thus we may define

$$\boxed{W_{a,b,\chi} = W_{a,b} \otimes \chi^{1/a} \text{ for } \chi \rightarrow A \text{ of degree } 0}$$

[iii] in $g=1$ this is Atiyah's classification

$$\begin{aligned} \mu: A \times A &\rightarrow A \times A \\ (x,y) &\rightarrow ax+by, x-y \end{aligned}$$

[iv] $\widehat{W}_{a,b} \cong W_{b,a}^\vee$ this gives the inductive construction of the W 's.

$$W_{a,b} \otimes \Theta \cong W_{a,a+b}$$

[v] $H^0(W_{a,b} \otimes \Theta)$ and $H^0(W_{a,b} \otimes \Theta)$ are naturally dual (for $a=b=1$ Wirtzinger)

3 Relationship with Verlinde bundle. If $(r, k) = 1$,

4/6

Example $E_{r,k} \cong \bigoplus W_{r,k}$ This is clear after pullback by $r: A \rightarrow A$; to see it before, it suffices to consider the action of $A[h]$ on $H^0(SU_r, \mathcal{I}^k)$ and show that it decomposes as sum of the same irred representation:

Theorem Let h be odd.

$$(*) \quad E_{hr, hk} = \bigoplus W_{r,k, \delta}^{m_\delta(r,k)} \quad \text{where}$$

$$[2] \quad m_\delta(r,k) = m_\delta(k,r) \quad [22] \quad \widehat{W}_{r,k, \delta} \cong W_{k,r, \delta} \Rightarrow E_{hr, hk} \cong E_{hk, hr}^v$$

Moreover

$$m_\delta(r,k) = \frac{1}{(r+k)^{2g}} \sum_{\delta | h} \frac{1}{\delta^{2g}} \underbrace{\left\{ \frac{h/\omega}{h/\delta} \right\}_g}_{\text{arithm. function}} \underbrace{\left[\frac{V_{\frac{h}{\delta}(g-1)+1}(r\delta, k\delta)}{\delta} \right]}_{\text{Verlinde #'s.}}$$

$$\left\{ \frac{\lambda}{h} \right\}_g = \begin{cases} 0, p_1^{a_1-1} \dots p_n^{a_n-1} + \lambda \\ \prod (\epsilon_i - \frac{1}{p_i^{2g}}) \text{ if } p_i^{a_i-1} + \epsilon_i \end{cases}$$

Corollary $H^0(U_X(r), \mathbb{H}^k \otimes \det^*(\mathbb{H}))$ and $H^0(U_X(k), \mathbb{H}^r \otimes \det^*(\mathbb{H}))$ are naturally dual. \downarrow
 $H^0(A, E_{r,k} \otimes \mathbb{H}) \cong H^0(A, E_{k,r} \otimes \mathbb{H})$.

4 Action of torsion points on the space of generalized theta functions

(*) is obvious after pulling back by $hr: A \rightarrow A$. To prove it in general we need to understand the pullback $A[h]$ -equivariantly. To give the flavor of the proof, let $\alpha \in A[h]$ be a torsion point of order δ .

Then $\alpha: SU_X(hr) \rightarrow SU_X(hr), E \rightarrow E \otimes \alpha$. This action lifts to \mathcal{I}^{hk} which descends to $[SU_X(hr)/A[h]]$ so it has a natural $A[h]$ -lift. (uses h odd).

Theorem $\text{Trace}(\alpha | H^0(SU_X(hr), \mathcal{I}^{hk})) = \left(\frac{r}{r+k} \right)^g \sqrt{(g-1)\delta+1} \left(\frac{hr}{\delta}, \frac{hk}{\delta} \right)$

Beauville for $\delta = h, r = 1$ & Ramanan - Narasimhan.

$h = 2, r = 1$ / Andersen - Masbaum.

\downarrow
 need to look at action of α over fibers of fixed points.

Idea. $Y \xrightarrow{\pi} X$ is an étale cover of order δ , $g_Y = (g-1)\delta + 1$. This 5/6 determines $\pi_*: \mathcal{U}_Y(\frac{r}{\delta}) \rightarrow \mathcal{U}_X(\frac{r}{\delta})$ such that the α -fixed locus is the image of Γ_* . To compute the trace use Lefschetz-Riemann-Roch & nice cancellations.

Outlook: defined Verlinde bundles over Jacobians, expressed them as sums of building blocks which satisfy level-rank duality & interchanged by Fourier Mukai; determined the space of generalized theta functions as $A[\hbar]$ -module.

5 Strangeness duality over abelian surfaces $-X(V \otimes W) = \langle V, W \rangle = \int v_2 w_2 - v_0 w_4 - v_4 w_0$

M_V = moduli of sheaves with fixed topological type V , polarization H = generic $g_1(N) = g(V) - 1$
 $\downarrow \alpha^+, \alpha^-$, $\alpha^+ = \det E, \alpha^- = \det O(\widehat{E}_{det})^{-1}$ The fiber is K_V and we have $v(E) = \text{rank } r + c_1(E) + \chi(E)c_2$
 $\widehat{A} \times A$ [hol. sympl.]

natural $(\mathbb{H})_W \rightleftharpoons M_V$ line bundles indexed by W . where $\langle V, W \rangle = 0$:
 $(\mathbb{H})_W = \det pr_* (\mathcal{F} \otimes \phi^* W)^{-1}$ where $\mathcal{F} \rightarrow M_V \times A$ universal

It is expected that

$H^0(M_V, (\mathbb{H})_W)$ and $H^0(K_W, (\mathbb{H})_V)$ are naturally dual. (equality of dimensions = checked). The method of Verlinde bundles is expected to give results.

$$H^0(M_V^+, (\mathbb{H})_W) \text{ and } H^0(M_W^+, (\mathbb{H})_V) = \frac{1}{2} \frac{g(VW)^2}{d_V + d_W} \binom{d_V + d_W}{d_V}$$

$$H^0(M_W^+, (\mathbb{H})_V) \text{ and } H^0(M_V^-, (\mathbb{H})_W) = \frac{1}{2} \frac{g(VW)^2}{d_V + d_W} \binom{d_V + d_W}{d_V}$$

→ the true Verlinde bundle remains a mystery. but could provide an

approach to this problem

$$\downarrow = \frac{d_V^2}{d_V + d_W} \binom{d_V + d_W}{d_V}, d_V = \frac{1}{2} \langle V, V \rangle$$

$\boxed{\dim M_V = 2d_V + 2}$

Assumptions about v : $-v$ is a primitive elt. of lattice 6/6
 $-rk(v) > 0$ or $rk(v) = 0$ & $c_1(v)$ effective, $\chi(v) \neq 0$.

Example ($g=1$), $\mathbb{E}_{h,h(g-1)} = \mathbb{H}^{g-1} \otimes \left(\sum_{S=h\text{-torsion}} L_S^{\oplus n} \oplus \mathcal{O}^{2+1} \right)$, $h = \text{prime}$

$$n = \frac{1}{h^2} \left(\frac{1}{g} \binom{hg}{h} - 1 \right)$$

Atiyah bundles $0 \rightarrow W_{h_1, d_1} \rightarrow W_{h, d} \rightarrow W_{h_2, d_2} \rightarrow 0$ $0 \leq \frac{d_1}{h_1} < \frac{d_2}{h_2} < 1$

The Verlinde bundles
in higher genus

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X smooth complex projective curve $g \geq 1$

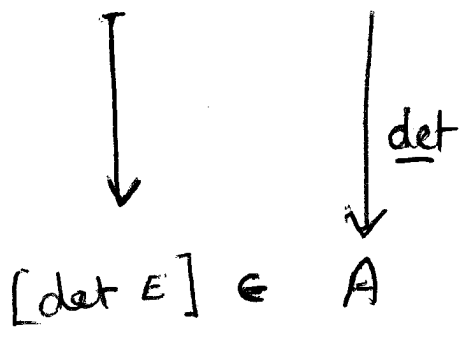
$A = \text{Jac}^\circ(X)$ - has principal polarization Θ

Let K be the theta characteristic

$$\Theta = \{ L : h^0(L \otimes K) \neq 0 \}$$

Let the rank $r \geq 1$, and let
of bundles in moduli

$[E] \in \mathcal{U}_X(r) = \text{mod. space of semistable rank } r, \text{ deg } 0$
bundles on X .



Theta divisors do exist on $\mathcal{U}_X(r)$.

$$\Theta_K = \{ E : h^0(E \otimes K) \neq 0 \}$$

primitive class in $\mathcal{U}_X(r)$ twist by K to get sections.

Θ_K corresponds to line bundles.

M. Popa :

$$\mathbb{E}_{r,k} = \text{det}^* \Theta^k := \text{Verlinde bundle for rank } r, \text{ level } k$$

Goal : describe $\mathbb{E}_{r,k}$.

- What is the rank of $\mathbb{E}_{r,k}$?

Verlinde formula

- Fiber of $\det^{-1}(0) = \text{rank } r \text{ bundles}$
with trivial determinant $= \text{SU}_X(r)$.
- Θ_X 's restrict to $\text{SU}_X(r)$
→ ample generator \mathcal{L} of $\text{Pic}(\text{SU}_X(r))$.
- $\mathbb{E}_{r,k}|_0 = h^0(\text{SU}_X(r), \mathcal{L}^k)$
= generalized theta fn of rank r , level k

$$h^0(\text{SU}_X(r), \mathcal{L}^k) = \left(\frac{r}{r+k}\right)^g v_g(r, k)$$

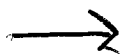
where $v_g(r, k) = (r+k)^{r(g-1)} \sum_{\substack{S \subseteq \{1, \dots, r+k\} \\ \#S = r}} \prod_{s \neq t, s, t \in S} \left| 2 \sin \frac{\pi(s-t)}{r+k} \right|^{1-g}$

Level-rank symmetry:

$$v_g(r, k) = v_g(k, r)$$

$$[S \xleftrightarrow{\text{replace with}} S^c]$$

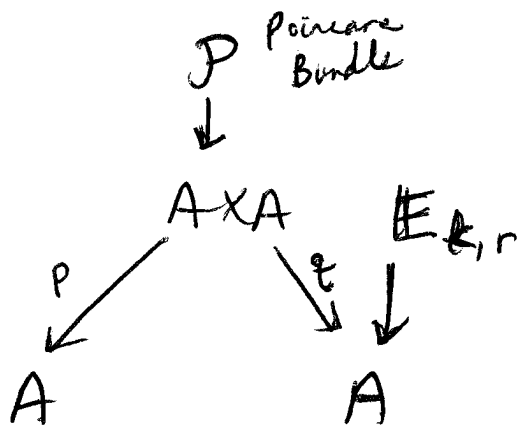
Geometric explanation of this symmetry:
(strange duality theorem)



Thm \exists geometrically induced isomorphism between the following two sections:

$$H^0(\mathcal{U}_X(k), \theta^r)^\vee \rightarrow H^0(\mathrm{SU}_X(r), \mathcal{L}^k)$$

Fourier-Mukai transform \xrightarrow{M} Popa $\hat{E}_{k,r} \cong E_{r,k}^\vee$ reinterpret Strange Duality Theorem



$$R\mathcal{P}_* (q^* E_{k,r} \otimes \mathcal{P})$$

- Look at the fiber over the origin $\{0\}$.
- LHS becomes $h^0(E_{k,r}) \cong h^0(\mathcal{U}_X(k), \theta^r)$
- RHS becomes $h^0(\mathrm{SU}_X(r), \mathcal{L}^k)^\vee$

- [2] Vector bundles on (A, θ) , $\dim A = g$
 θ symmetric $(-1)^* \theta \cong \theta$

Semihomogeneous vector bundles

We say v. bundle W is s.h. if
 $\forall x \in A \quad \exists y \in \hat{A}, t_x^* W \cong W \otimes y$

Example [i]. line bundles

- [ii] \exists isogeny f s.t. $f^* W \cong \bigoplus$ ^{the same} line bundle
 [iii] The stable ones move in a g -dim'l moduli space.
 [iv] $ch(W) = rk(W) \cdot \exp\left(\frac{c_1(W)}{rk(W)}\right)$

Prop. For any odd coprime integers (a, b) ,
 $\exists!$ ^{simple} semihomogeneous $W_{a,b}$ s.t.
 $rank W_{a,b} = a^g$
 $\det W_{a,b} = \theta^{a^{g-1}b}$
 and $W_{a,b}$ is symmetric.

Remarks [i]. $g=1$, Atiyah's thm ^{$b \geq 0$}
 [ii] $a^* W_{a,b} \cong \bigoplus \theta^{ab}$, $\underline{a}: A \rightarrow A$, ^{multn. by \underline{a}}
 [iii] $\overline{W}_{a,b} \cong \widehat{W}_{b,a}$
~~[iv] $W_{a,b} \cong W_{b,a}$~~

Set $W_{a,b,y} = W_{a,b} \otimes y^{\gamma_a}$, $y = \deg 0$ line bundle.

Thm. (assuming odd integers)

If $(r, k) = 1$,

$$E_{hr, hk} \cong \bigoplus W_{r, k, y}^{\oplus m_y(r, k)}$$

$y \rightarrow h$ -torsion
line bundle on A

where $\boxed{m_y(r, k) = m_y(k, r)}$

$$m_y(r, k) = \frac{1}{(r+k)g} \sum_{\delta | h} c_g(\delta, \text{ord } y) \chi_{\frac{h}{\delta}(g-1)+1}^{(r\delta, k\delta)}$$

5/6
If $(r, k) = h$,
then change
 r and k so that
 $(r, k) = 1$.

[3.] Action of torsion points on generalized theta functions

$$\begin{array}{ccccc} \Theta^{hk} & \longrightarrow & U_X(hr) & \longleftarrow & SU_X(hr) \times A & t(E, L) = E \otimes L \\ & & \downarrow \det & & \downarrow pr & \\ & & A & \xleftarrow{hr} & A & \\ & & E_{hr, hk} & \longrightarrow & & \end{array}$$

$$t^* \Theta \cong \mathcal{L} \otimes \Theta^{hr}$$

$$(hr)^* E \cong \underbrace{H^0(SU(hr), \mathcal{L}^{hk})}_{\text{need to understand } \downarrow \text{ action of "hr-torsion points" on it.}} \otimes \Theta^{h^2kr}$$

Will discuss only h -torsion pts on X .



If $\alpha \in A[h]$, acts on $\text{su}_X(hr)$ by

$$E \rightarrow E \otimes \alpha$$

The action lifts to \mathcal{L}^{hk} .

Thm $\text{Trace}(\alpha | H^0(\text{su}_X(hr), \mathcal{L}^{hk}))$
 $= \left(\frac{r}{r+k}\right)^g \sum_{(g-1)\delta+1}^g \left(\frac{hr}{\delta}, \frac{hk}{\delta}\right),$
 where $\delta = \text{ord}(\alpha)$.

- Proved when $\delta = h$, $r=1$ by Beauville
- " " $h=2$, $r=1$, Anderson-Masbaum
- LRR computation
- The picture for curves has analogues for sheaves on surfaces ($K3$, abelians).