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The Verlinde bundles in higher genus

Two appologies: the moduli spaces studied here are rather classical; the word "modern" may not apply. The second is more serious: there seems to be two different Verlinde bundles in the literature; they live on different moduli spaces and have different flavors. I will only speak about one of them, to the disappointment of the audience.

$\boxed{1}$ Setup $\boxed{a}$ Let X = smooth complex projective curve of genus $g \geq 1$. Classical objects associated to X are:

- the Jacobian $A = \mathrm{Jac}^0(X) \ni [L]$ = degree 0 line bundle. (abelian variety)
- principal polarization: let $\kappa^2 = \omega_X$ be a theta characteristic. Define

$\Theta_\kappa = \{ L : h^0(L \otimes \kappa) \neq 0 \}$. is a divisor in A: Θ is symmetric $(-1)^*\Theta \cong \Theta$

- theta functions of level k: $H^0(A, \Theta^k)$ = dimension is k^g.

$\boxed{b}$ Higher rank picture: Replace A by

$\Theta \rightsquigarrow \mathbb{L} \boxtimes \Theta^r$

$U_X(r,0)$ = semistable vector bundles of rank r, $c_1 = 0$

$$\begin{tikzcd} U_X(r,0) \arrow[d, "\det"] & E \arrow[d] & SU(r) \times A \arrow[l, "t"'] \arrow[d, "\pi"] \\ A & \det E & A \arrow[l, "r"'] \end{tikzcd}$$

the fibration becomes trivial

Theta divisors $\Theta_\kappa = \{ E : h^0(E \otimes \kappa) \neq 0 \}$ and accordingly, theta functions of rank r and level k.

Variant of the construction: the fiber of $\det^{-1}(\mathcal{O})$: $SU_X(r, \mathcal{O})$. All Θ_κ's restrict to the same line bundle $\mathbb{L} \longrightarrow SU_X(r, \mathcal{O})$ which generates Picard.

$h^0(SU_X, \mathbb{L}^k)$ = rank r, level k, fixed trivial determinant theta's.

[c] Dimension: $h^0(SU_X(r), \mathbb{L}^k) = \left(\frac{r}{r+k}\right)^g V_g(r,k)$ where

$$V_g(r,k) = (r+k)^{r(g-1)} \sum_{\substack{S\subset[r+k] \\ \#S=r}} \prod_{s\neq t} \left| 2\sin \frac{s-t}{r+k}\right|^{1-g}.$$

Note: level-rank symmetry: $\boxed{V_g(r,k) = V_g(k,r).}$

For instance $V_g(r,1) = V_g(1,r) = (r+1)^g \Rightarrow h^0(SU_X(r), \mathbb{L}) = r^g =$
$=$ # level r theta functions. In fact:

[d] Strange duality:

Thm (Marian-Oprea, Belkale) There is an isomorphism:

$$H^0(\mathcal{U}_X(k), \Theta^r) \text{ and } H^0(SU_X, \mathbb{L}^k)^\vee \text{ induced geometrically}$$

by the equation of the divisor $\{(E,F) \mid h^0(E F K) \neq 0\}$.

[e] Verlinde bundles (Popa): $\mathbb{E}_{r,k} = \det_*(\Theta^{\otimes k})$ = bundles of generalized theta functions, fixed rank/level/ determinant but vary determinant in A

Corollary the Fourier-Mukai transform with respect to the Poincaré bundle $\mathcal{P}$ on $A\times A$ induces an isomorphism $(\mathcal{P}|_{A\times 0}, \mathcal{P}|_{0\times A} = \text{trivial})$

$$\underbrace{\widehat{\mathbb{E}_{k,r}}}_{IT_0} \cong \mathbb{E}_{r,k}^\vee \quad \text{where } \boxed{\wedge : \alpha \longrightarrow Rp_{1*}(q^*\alpha\cdot\mathcal{P}).}$$

Properties: (Popa) [1] $\mathbb{E}_{r,k}$ is ample, polystable, globally generated if $k\geq r+1$, IT_0 semihomogeneous. [2] $\boxed{r^*\mathbb{E}_{r,k} \cong \Theta^{rk} \otimes H^0(SU_X(r), \mathcal{L}^k)}$

Over the origin: $\cancel{\widehat{\mathbb{E}}_{k,k}|_0 \cong h^0(\mathbb{E}_{r,k}) \cong h^0(\mathcal{U}_r}$

$$\widehat{\mathbb{E}}_{k,r}\Big|_0 = h^0(\mathbb{E}_{k,r}) = h^0(\mathcal{U}_X(k), \Theta^r) = h^0(SU(r), \mathbb{L}^k)^\vee = \mathbb{E}_{r,k}$$

Today: describe $\boxed{\text{Verlinde bundles in full generality.}}$

Verlinde bundles have been used by Popa to study:

[i] positivity properties of the theta bundles on the moduli space $\mathcal{U}_X(r)$.

[ii] restriction to the curve can be used to give basepoints for $|\mathcal{L}|$ on SU_X.

– today we will describe the Verlinde bundles in full generality.

[I] Vector bundles on abelian varieties

[a] Let (A, Θ) be an abelian variety of dimension g, $(-1)^*\Theta = \Theta$ a symmetric polarization. Moduli spaces of vector bundles over A have been studied by Mukai, and a class of vector bundles emerged as particularly important.

Def W is semihomogeneous if $t_x^* W \cong W \otimes y$ for $x \in A, y \in \hat{A}$. (Mukai/Umemura)

Examples [i] line bundles

[ii] up to isogenies, W splits as sums of line bundles.

[iii] W moves in a g-dim'l moduli space, isogenous to A.

[iv] $\operatorname{ch} W = \operatorname{rk} W \cdot \exp\left(c_1(W)/\operatorname{rk} W\right)$. for slope $\mu = \frac{b\Theta}{a}$

Proposition For odd coprime integers, (a,b) there is a unique simple symmetric semihomogeneous $W_{a,b}$ with

$$\boxed{\operatorname{rank} W_{a,b} = a^g \text{ and } \det W_{a,b} = a^{g-1} b\,\Theta}$$

Moreover, [i] $a^* W_{a,b} \cong \bigoplus \Theta^{ab} \cong \Theta^{ab} \otimes a^g\text{-dim'l vector space}$

[ii] $W_{a,b} \otimes (a\text{-torsion}) \cong W_{a,b}$. Thus we may define

$$\boxed{W_{a,b,\chi} = W_{a,b} \otimes \chi^{1/a} \text{ for } \chi \to A \text{ of degree } 0}$$

[iii] in $g=1$ this is Atiyah's classification

$\mu : A \times A \longrightarrow A \times A$

$x, y \longrightarrow ax+by, x-y.$

[iv] $\widehat{W_{a,b}} \cong W_{b,a}^{\vee}$ this gives the inductive construction of the W's.

$W_{a,b} \otimes \Theta \cong W_{a,a+b}$

[v] $H^0(W_{a,b} \otimes \Theta)$ and $H^0(W_{a,b} \otimes \Theta)$ are naturally dual (for $a=b=1$ Wirtinger)

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3 <u>Relationship with Verlinde bundle.</u> If $(r,k)=1$,

<u>Example</u> $\boxed{\mathbb{E}_{r,k} \cong \bigoplus W_{r,k}}$ This is clear after pullback by $r: A \longrightarrow A$; to see it before, it suffices to consider the action of $A[r]$ on $H^0(SU_r, \mathcal{L}^k)$ and show that it decomposes as sum of the same irred representation:

<u>Theorem</u> Let h be odd.

$$(*) \quad \mathbb{E}_{hr,hk} = \bigoplus W_{r,k,\zeta}^{\,m_\zeta(r,k)} \quad \text{where}$$

1.2 $m_\zeta(r,k) = m_\zeta(k,r)$ 2.2 $\widehat{W}_{r,k,\zeta} \cong W^{\vee}_{k,r,\zeta'} \Rightarrow \mathbb{E}_{hr,hk} \cong \mathbb{E}^{\vee}_{hk,hr}$.

Moreover

$$m_\zeta(r,k) = \frac{1}{(r+k)^{2g}} \sum_{\delta | h} \frac{1}{\delta^{2g}} \underbrace{\left\{\frac{h/\omega}{h/\delta}\right\}_g}_{\text{arithm. function}} \boxed{V_{\frac{h}{\delta}(g-1)+1}(r\delta, k\delta)}$$

Verlinde #'s.

$$\left\{\frac{\lambda}{h}\right\}_g = \begin{cases} 0, & p_1^{a_1-1}\cdots p_n^{a_n-1} \nmid \lambda \\ \prod\left(\epsilon_i - \frac{1}{p_i^{2g}}\right) & \text{if } p_i^{a_i-1+\epsilon_i} \| \lambda. \end{cases}$$

<u>Corollary</u> $H^0(\mathcal{U}_X(r), \Theta^k \otimes \det^*\Theta)$ and $H^0(\mathcal{U}_X(k), \Theta^r \otimes \det^*\Theta)$ are naturally dual.

$$\begin{array}{ccc} H^0(\mathcal{U}_X(r), \Theta^k \otimes \det^*\Theta) & & H^0(\mathcal{U}_X(k), \Theta^r \otimes \det^*\Theta) \\ \wr| & & \| \\ H^0(A, \mathbb{E}_{r,k} \otimes \Theta) & \cong & H^0(A, \mathbb{E}_{k,r} \otimes \Theta). \end{array}$$

4 <u>Action of torsion points on the space of generalized theta functions</u>

$(*)$ is obvious after pulling back by $hr: A \longrightarrow A$. To prove it in general we need to understand the pullback $A[hr]$-equivariantly. To give the flavor of the proof, let $\alpha \in A[h]$ be a torsion point of order δ.

Then $\alpha: SU_X(hr) \longrightarrow SU_X(hr)$, $E \longrightarrow E \otimes \alpha$. This action lifts to $\mathcal{L}^{hk}$ which descends to $\boxed{SU_X(hr)/A[h]}$ so it has a natural $A[h]$-lift (uses h odd).

<u>Theorem</u> $$\boxed{\operatorname{Trace}\left(\alpha \,\middle|\, H^0(SU_X(hr), \mathcal{L}^{hk})\right) = \left(\frac{r}{r+k}\right)^g V_{(g-1)\delta+1}\left(\frac{hr}{\delta}, \frac{hk}{\delta}\right)}$$

need to look at action of α over fibers of fixed points.

Beauville for $\delta = h$, $r=1$ & Ramanan – Narasimhan.

$h=2$, $r=1$ / Andersen – Masbaum.

Idea. $Y \xrightarrow{\pi} X$ is an étale cover of order δ, $g_Y = (g-1)\delta + 1$. This determines $\pi_*: \mathcal{U}_Y\left(\frac{hr}{\delta}\right) \longrightarrow \mathcal{U}_X(hr)$. such that the α-fixed locus is the image of π_*. To compute the trace use Lefschetz - Riemann - Roch & nice cancelations.

Outlook: defined Verlinde bundles over Jacobians, expressed them as sums of building blocks which satisfy level-rank duality & interchanged by Fourier Mukai; determined the space of generalized theta functions as $A[h]$-module.

5 Strange duality over abelian surfaces

$$-\chi(v \otimes w) = \langle v, w \rangle = \int v_2 w_2 - v_0 w_4 - v_4 w_0$$

$$v(E) = r + c_1(E) + \chi(E)\omega$$

M_v = moduli of sheaves with fixed topological type v, polarization H = generic

$c_1(\Lambda) = c_1(v)$

$$\begin{array}{c} M_v \\ \downarrow \alpha^+, \alpha^- \\ \hat{A} \times A \end{array}, \quad \boxed{\alpha^+ = \det E, \ \alpha^- = \det \circ (\widehat{E \det \Lambda^{-1}})}$$

The fiber is K_v. and we have $\boxed{\text{hol. sympl.}}$

natural $\Theta_w \longrightarrow M_v$ line bundles indexed by w. where with $\langle v, w \rangle = 0$:

$$\boxed{\Theta_w = \det pr_* (\mathcal{F} \otimes p^* w)^{-1}.}$$

where $\mathcal{F} \xrightarrow{\text{universal}} M_v \times A$

It is expected that

$H^0(M_v, \Theta_w)$ and $H^0(K_w, \Theta_v)$ are naturally dual. (equality of dimensions = checked). The method of Verlinde bundles is expected to give results.

$$H^0(M^+_v, \Theta_w) \text{ and } H^0(M^+_w, \Theta_v) = \frac{1}{2} \frac{c_1(vw)^2}{d_v + d_w} \binom{d_v + d_w}{d_v}$$

$$H^0(M^+_w, \Theta_v) \text{ and } H^0(M^-_w, \Theta_v) = \frac{1}{2} \frac{c_1(v\hat{w})^2}{d_v + d_w} \binom{d_v + d_w}{d_v}$$

– the true Verlinde bundle remains a mistery. but could provide an approach to this problem

$$= \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}, \quad d_v = \frac{1}{2}\langle v, v \rangle.$$

$$\boxed{\dim M_v = 2d_v + 2}$$

Assumptions about v : – v is a primitive elt. of lattice

– $rk(v) > 0$ or $rk(v) = 0$ & $c_1(v)$ effective, $\chi(v) \neq 0$.

Example $(g=1)$, $\mathbb{E}_{h, h(g-1)} = \Theta^{g-1} \otimes \left(\sum_{\zeta = h\text{-torsion}} L_\zeta^{\oplus n} \oplus \mathcal{O}^{n+1} \right)$, h = prime

$$\boxed{n = \frac{1}{h^2}\left(\frac{1}{g}\binom{hg}{h} - 1\right)}$$

Atiyah bundles $0 \longrightarrow W_{h_1, d_1} \longrightarrow W_{h,d} \longrightarrow W_{h_2, d_2} \longrightarrow 0$ $\quad 0 \leq \frac{d_1}{h_1} < \frac{d_2}{h_2} < 1$

The Verlinde bundles in higher genus

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X smooth complex projective curve $g \geq 1$

$A = \mathrm{Jac}^0(X)$ - has principal polarization Θ
Let κ be the theta characteristic
$\Theta = \{ L : h^0(L \otimes \kappa) \neq 0 \}$

Let the rank $r \geq 1$, and let (of bundles in moduli)

$$\begin{array}{ccl} [E] & \in & \mathcal{U}_X(r) = \text{mod. space of semistable rank } r, \deg 0 \text{ bundles on } X. \\ \big\downarrow & & \big\downarrow \underline{\det} \\ [\det E] & \in & A \end{array}$$

Theta divisors do exist on $\mathcal{U}_X(r)$.

primitive class
$\Theta_\kappa = \{ E : h^0(E \otimes \kappa) \neq 0 \}$ twist by κ to get sections.
in $\mathcal{U}_X(r)$
Θ_κ corresponds to line bundles.

M. Popa:
$\mathbb{E}_{r,k} = \underline{\det}_* \Theta^k :=$ Verlinde bundle for rank r, level k

Goal: describe $\mathbb{E}_{r,k}$.

- What is the rank of $\mathbb{E}_{r,k}$?

<u>Verlinde formula</u>

- fiber of $\det^{-1}(\mathcal{O})$ = rank r bundles with trivial determinant $= SU_X(r)$.
- $\Theta_{\mathbb{K}}$'s restrict to $SU_X(r)$

$\rightarrow$ ample generator $\mathcal{L}$ of $Pic(SU_X(r))$.

- $\mathbb{E}_{r,k}|_{\mathcal{O}} = h^0(SU_X(r), \mathcal{L}^k)$

= generalized theta fns of rank r, level k

$$h^0(SU_X(r), \mathcal{L}^k) = \left(\frac{r}{r+k}\right)^g v_g(r,k)$$

where $v_g(r,k) = (r+k)^{r(g-1)} \sum_{\substack{S \subseteq \{1,\dots,r+k\} \\ \#S = r}} \prod_{s \neq t,\, s \in S,\, t \notin S} \left| 2 \sin \frac{\pi(s-t)}{r+k} \right|^{1-g}$

Level-rank symmetry:

$$v_g(r,k) = v_g(k,r)$$

$$\Big[\, S \xleftrightarrow[\text{replace with}]{} S^c \,\Big]$$

Geometric explaination of this symmetry:
(strange duality theorem)

$\longrightarrow$

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Thm $\exists$ geometrically induced isomorphism between the following two sections:

$$H^0(\mathcal{U}_X(k), \theta^r)^\vee \longrightarrow H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$$

M Popa reinterpret Strange Duality Theorem

Fourier-Mukai transform $\rightarrow$ $\widehat{\mathbb{E}}_{k,r} \cong \mathbb{E}_{r,k}^\vee$

$\mathcal{P}$ Poincare Bundle

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d] & \\
 & A \times A \arrow[dl, "p"'] \arrow[dr, "q"] & \mathbb{E}_{k,r} \arrow[d] \\
A & & A
\end{tikzcd}$$

$$Rp_*(q^* \mathbb{E}_{k,r} \otimes \mathcal{P})$$

- Look at the fiber over the origin $\{0\}$.
- LHS becomes
  $h^0(\mathbb{E}_{k,r}) \cong h^0(\mathcal{U}_X(k), \theta^r)$
- RHS becomes
  $h^0(SU_X(r), \mathcal{L}^k)^\vee$

$\boxed{2}$ Vector bundles on (A, θ), $\dim A = g$
θ symmetric $(-1)^* \theta \cong \theta$

Semihomogeneous vector bundles

We say v. bundle W is s.h. if
$\forall x \in A \quad \exists y \in \hat{A}$, $t_x^* W \cong W \otimes y$.

Example $\boxed{i}$. line bundles
$\boxed{ii}$. $\exists$ isogeny f s.t. $f^* W \cong \bigoplus$ the same line bdle
$\boxed{iii}$ The stable ones move in a g-dim'l moduli space.
$\boxed{iv}$ $ch(W) = rk(W) \cdot \exp\left(\frac{c_1(W)}{rk(W)}\right)$

Prop. For any odd coprime integers (a,b),
$\exists !$ simple semihomogeneous $W_{a,b}$ s.t.
rank $W_{a,b} = a^g$
$\det W_{a,b} = \theta^{a^{g-1} b}$
and $W_{a,b}$ is symmetric.

Remarks $\boxed{i}$. $g=1$, Atiyah's thm
$\boxed{ii}$ $\underline{a}^* W_{a,b} \cong \bigoplus \theta^{ab}$, $\underline{a}: A \to A$, multn. by $\underline{a}$, $b>0$
$\boxed{iii}$ $W_{a,b}^\vee \cong \hat{W}_{b,a}$.

Set $W_{a,b,y} = W_{a,b} \otimes y^{1/a}$, y = deg 0 line bundle.

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If $(r,k)=h$, then change r and k so that $(r,k)=1$.

<u>Thm.</u> (assuming odd integers).
If $(r,k)=1$,
$$\mathbb{E}_{hr,hk} \cong \bigoplus W_{r,k,y}^{\oplus m_y(r,k)}$$

$y \rightsquigarrow$ h-torsion line bundle on A.

where $\boxed{m_y(r,k) = m_y(k,r)}$

$$m_y(r,k) = \frac{1}{(r+k)^g} \sum_{\delta \mid h} c_g(\delta, \operatorname{ord} y)\, V^{(r\delta, k\delta)}_{\frac{h}{\delta}(g-1)+1}$$

$\boxed{3.}$ Action of torsion points on generalized theta functions

$$\begin{tikzcd} \Theta^{hk} \arrow[r] \arrow[d] & \mathcal{U}_X(hr) \arrow[d, "\underline{\det}"'] & SU_X(hr) \times A \arrow[l] \arrow[d, "pr"] \\ \mathbb{E}_{hr,hk} \arrow[r] & A & A \arrow[l, "hr"'] \end{tikzcd}$$

$t(E,L) = E \otimes L$

$$t^* \theta \cong \mathcal{L} \boxtimes \theta^{hr}$$

$$(hr)^* \mathbb{E} \cong \underbrace{H^0(su(hr), \mathcal{L}^{hk})}_{\substack{\text{need to understand + action of} \\ \text{"}hr\text{-torsion points" on it.}}} \otimes \theta^{h^2kr}$$

Will discuss only h-torsion pts on X. $\to$

If $\alpha \in A[h]$, acts on $su_X(hr)$ by

$$E \longrightarrow E \otimes \alpha.$$

The action lifts to $\mathcal{L}^{hk}$.

<u>Thm</u> Trace$\left(\alpha \mid H^0(su_X(hr), \mathcal{L}^{hk})\right)$

$$= \left(\frac{r}{r+k}\right)^g \vartheta_{(g-1)\delta+1}\left(\frac{hr}{\delta}, \frac{hk}{\delta}\right),$$

where $\delta = \mathrm{ord}(\alpha)$.

- Proved when $\delta = h$, $r=1$ by Beauville
- " " $h=2$, $r=1$, Anderson–Masbaum
- LRR computation
- The picture for curves has analogues for sheaves on surfaces (K3, <u>abelians</u>).