LOG CANONICAL SINGULARITIES ARE DU BOIS

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1. INTRODUCTION

A recurring difficulty in the Minimal Model Program (MMP) is that while log terminal singularities are quite well behaved (for instance, they are rational), log canonical singularities are much more complicated; they need not even be Cohen-Macaulay. The aim of this paper is to prove that, as conjectured in [Kol92, 1.13], log canonical singularities are Du Bois. The concept of Du Bois singularities, introduced by [Ste83], is a weakening of rationality. It is not known how to define Du Bois singularities in positive characteristic, so we work over a field of characteristic 0 throughout the paper. The precise definition is rather involved, see (1.9), but our applications rely only on the following consequence:

Corollary 1.1. Let $X$ be a proper scheme of finite type over $\mathbb{C}$. If $(X, \Delta)$ is log canonical for some $\mathbb{Q}$-divisor $\Delta$, then the natural map

$$H^i(X^{an}, \mathbb{C}) \to H^i(X^{an}, \mathcal{O}_{X^{an}}) \cong H^i(X, \mathcal{O}_X)$$

is surjective for all $i$.

Using [DJ74, Lemme 1] this implies the following:

Corollary 1.2. Let $\phi: X \to B$ be a proper, flat morphism of complex varieties with $B$ connected. Assume that for all $b \in B$ there exists a $\mathbb{Q}$-divisor $D_b$ on $X_b$ such that $(X_b, D_b)$ is log canonical. Then $h^i(X_b, \mathcal{O}_{X_b})$ is independent of $b \in B$ for all $i$.

Notice that we do not require that the divisors $D_b$ form a family.

We also prove flatness of the cohomology sheaves of the relative dualizing complex of a projective family of log canonical varieties (1.7). Combining this result with a Serre duality type criterion (7.9) gives another invariance property:

Corollary 1.3. Let $\phi: X \to B$ be a flat, projective morphism, $B$ connected. Assume that for all $b \in B$ there exists a $\mathbb{Q}$-divisor $D_b$ on $X_b$ such that $(X_b, D_b)$ is log canonical.

Then, if one fiber of $\phi$ is Cohen-Macaulay (resp. $S_k$ for some $k$), then all fibers are Cohen-Macaulay (resp. $S_k$).

Remark 1.3.1. The $S_k$ case of this result answers a question posed to us by Valery Alexeev and Christopher Hacon. For arbitrary flat, proper morphisms, the set of fibers that are Cohen-Macaulay (resp. $S_k$) is open, but not necessarily closed. Thus the key point of (1.3) is to show that this set is also closed.

Date: February 4, 2009.

2000 Mathematics Subject Classification. 14J17, 14B07, 14E30, 14D99.

János Kollár was supported in part by NSF Grant DMS-0758275.

Sándor Kovács was supported in part by NSF Grant DMS-0554697 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.
The generalization of these results to the semi log canonical case turns out to be easy, but it needs some foundational work which will be presented elsewhere. The general case then implies that each connected component of the moduli space of stable log varieties parametrizes either only Cohen-Macaulay or only non-Cohen-Macaulay objects.

The main theorem of the paper is the following:

**Theorem 1.4.** Let \( f : Y \to X \) be a proper morphism with connected fibers between normal varieties. Assume that there exists an effective \( \mathbb{Q} \)-divisor on \( Y \) such that \((Y, \Delta)\) is lc and \( K_Y + \Delta \sim_{\mathbb{Q},f} 0 \). Then \( X \) is DB.

More generally, let \( W \subset Y \) be a reduced, closed subscheme that is a union of log canonical centers of \((Y, \Delta)\). Then \( f(W) \subset X \) is DB.

Notice that the conditions of the theorem are satisfied if \((X, \Delta)\) is log canonical and we choose \( Y = X \) and \( f = \text{id}_X \). Thus log canonical singularities are Du Bois. For earlier results related to this conjecture of [Kol92, 1.13], see [Kol95, §12], [Kov99, Kov00b, KSS08]. The above, more general, form of (1.4) is better suited to working with log canonical centers.

There are three, more technical results that should be of independent interest. The first is a quite flexible criterion for Du Bois singularities:

**Theorem 1.5.** Let \( f : Y \to X \) be a proper morphism between reduced schemes of finite type over \( \mathbb{C} \), \( W \subset X \) an arbitrary subscheme, and \( F := f^{-1}(W) \), equipped with the induced reduced subscheme structure. Assume that the natural map \( \varrho : I_{W \subset X} \to Rf_*I_{F \subset Y} \)

admits a left inverse \( \varrho' \). Then if \( Y, F, \) and \( W \) all have DB singularities, then so does \( X \).

**Remark 1.5.1.** Notice that we do not require \( f \) to be birational. On the other hand the assumptions of the theorem and [Kov00a, Thm 1] imply that if \( Y \setminus F \) has rational singularities, e.g., if \( Y \) is smooth, then \( X \setminus W \) has rational singularities as well.

The second is a variant of the connectedness theorem [Kol92, 17.4] for not necessarily birational morphisms.

**Theorem 1.6.** Let \( f : Y \to X \) be a proper morphism with connected fibers between normal varieties. Assume that \((Y, \Delta)\) is lc and \( K_Y + \Delta \sim_{\mathbb{Q},f} 0 \). Let \( Z_1, Z_2 \subset Y \) be lc centers of \((Y, \Delta)\). Then every irreducible component of \( Z_1 \cap f^{-1}(f(Z_2)) \) is also an lc center of \((Y, \Delta)\).

The third is the flatness of the cohomology sheaves of the relative dualizing complex of a DB morphism:

**Theorem 1.7.** Let \( \phi : X \to B \) be a flat projective morphism such that all fibers are Du Bois. Then the cohomology sheaves \( h^i(\omega^*_\phi) \) are flat over \( B \), where \( \omega^*_\phi \) denotes the relative dualizing complex of \( \phi \).

**Definitions and Notation 1.8.** Let \( K \) be an algebraically closed field of characteristic 0. Unless otherwise stated, all objects are assumed to be defined over \( K \), all schemes are assumed to be of finite type over \( K \) and a morphism means a morphism between schemes of finite type over \( K \).

A pair \((X, \Delta)\) consists of a variety \( X \) and an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( X \). If \((X, \Delta)\) is a pair, then \( \Delta \) is called a boundary if \( [(1 - \varepsilon)\Delta] = 0 \) for all \( 0 < \varepsilon < 1 \), i.e., the coefficients of all
irreducible components of $\Delta$ are in the interval $[0, 1]$. For the definition of klt, dlt, and lc pairs see [KM93]. If $(X, \Delta)$ is a pair, then a minimal dlt model of $(X, \Delta)$ is a dlt pair $(X^m, \Delta^m)$ together with a birational morphism $f^m : (X^m, \Delta^m) \to (X, \Delta)$ such that $f^m_* \Delta^m = \Delta$ and the discrepancy of every exceptional divisor of $f^m$ is at most $-1$.

For morphisms $\phi : X \to B$ and $\vartheta : T \to B$, the symbol $X_T$ will denote $X \times_B T$ and $\vartheta_T : X_T \to T$ the induced morphism. In particular, for $b \in B$ we write $X_b = \phi^{-1}(b)$. Of course, by symmetry, we also have the notation $\vartheta_X : T_X \simeq X_T \to X$ and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}_T$ will denote the $\mathcal{O}_{X_T}$-module $\vartheta_X^* \mathcal{F}$.

For a morphism $\phi : X \to B$, the relative dualizing complex of $\phi$ (if it exists) will be denoted by $\omega^*_\phi$. In particular, for a (quasi-projective) scheme $X$, the dualizing complex of $X$ will be denoted by $\omega^*_X$.

For a subscheme $W \subseteq X$, the ideal sheaf of $W$ is denoted by $\mathcal{J}_{W \subseteq X}$ or if no confusion is likely, then simply by $\mathcal{J}_W$. For a point $x \in X$, $\kappa(x)$ denotes the residue field of $\mathcal{O}_{X,x}$.

The symbol $\simeq$ will mean isomorphism in the appropriate category. In particular, between complexes considered as objects in a derived category it stands for a quasi-isomorphism.

**Definition 1.9.** Consider a complex algebraic variety $X$. If $X$ is smooth and projective, its De Rham complex plays a fundamental role in understanding the geometry of $X$. When $X$ is singular, an analog of the De Rham complex, introduced by Du Bois, plays a similar role.

Let $X$ be a complex scheme of finite type. Based on Deligne’s theory of mixed Hodge structures, Du Bois defined a filtered complex of $\mathcal{O}_X$-modules, denoted by $\Omega^*_X$, that agrees with the algebraic De Rham complex in a neighborhood of each smooth point and, like the De Rham complex on smooth varieties, its analytization provides a resolution of the sheaf of locally constant functions on $X$ [DB81].

Du Bois observed that an important class of singularities are those for which $\Omega^0_X$, the zeroth graded piece of the filtered complex $\Omega^*_X$, takes a particularly simple form. He pointed out that singularities satisfying this condition enjoy some of the nice Hodge-theoretic properties of smooth varieties cf. (7.4). These singularities were christened Du Bois singularities by Steenbrink [Ste83]. We will refer to them as DB singularities and a variety with only DB singularities will be called DB.

The construction of the Du Bois complex $\Omega^*_X$ is highly non-trivial, so we will not include it here. For a thorough treatment the interested reader should consult [PS08] II.7.3. For alternative definitions, sufficient and equivalent criteria for DB singularities see [Kov99, Sch07, KSS08].

**Remark 1.10.** Recall that the seminormalization of $\mathcal{O}_X$ is $h^0(\Omega^0_X)$, the 0th cohomology sheaf of the complex $\Omega^*_X$, and so $X$ is seminormal if and only $\mathcal{O}_X \simeq h^0(\Omega^0_X)$ by [Sai00, 5.2] (cf. [Sch06, 5.4.17] and [Sch07, 4.8]). In particular, this implies that DB singularities are seminormal.

**Acknowledgments.** We would like to thank Valery Alexeev, Christopher Hacon and Karl Schwede for useful comments and discussions that we have benefited from. The otherwise unpublished Theorem 3.1 was communicated to us by Christopher Hacon. We are also grateful to Stefan Schröer for letting us know about Example 7.12.
2. A criterion for Du Bois singularities

In order to prove (1.5) we first need the following abstract derived category statement.

**Lemma 2.1.** Let $A, B, C, A', B', C'$ be objects in a derived category and assume that there exists a commutative diagram in which the rows form exact triangles:

\[(2.1.1)\]

Then there exists an exact triangle:

\[(2.1.2)\]

and a natural map $\delta : B \to D$.

Furthermore, let $\lambda$ denote the composition $\lambda : D \to B' \oplus_C 0 \oplus \text{id}_C$. Then $\lambda \circ \delta = \psi$ and $\alpha$ admits a left inverse if and only if $\delta$ admits one, $\delta' : D \to B$ such that $\psi \circ \delta' = \lambda$, and $\alpha$ is an isomorphism if and only if $\delta$ is.

**Proof.** Let $\eta : B' \oplus C \to C'$ be the natural map induced by $-\psi'$ on $B'$ and $\gamma$ on $C$, and $D$ the object that completes $\eta$ to an exact triangle as in (2.1.2).

Next consider the following diagram:

\[
\begin{array}{c}
\text{A'[1]} \\
\downarrow \\
\text{C} \\
\downarrow \\
\text{B'} \\
\downarrow \\
\text{B' \oplus C} \end{array} \quad \begin{array}{c}
\uparrow \\
\exists \vartheta \uparrow \\
\downarrow \\
\downarrow \\
\uparrow \end{array} \quad \begin{array}{c}
\text{D[1]} \\
\downarrow \\
\text{C'} \\
\downarrow \\
\text{0_{B'} \oplus \text{id}_C} \\
\downarrow \\
\text{(-} \text{id}_{B'}, 0) \end{array}
\]

The bottom triangle is a commutative triangle with the maps indicated. The triangles with one edge common with the bottom triangle are exact triangles with the obvious maps. Then by the octahedral axiom, the maps in the top triangle denoted by the broken arrows exist and form an exact triangle.

Observe that it follows that the induced map $\vartheta : C \to A'[1]$ agrees with $\zeta' \circ (\eta|_C) = \zeta' \circ \gamma$ which in turn equals $\alpha[1] \circ \zeta$ by (2.1.1).
Therefore one has the following commutative diagram where the rows form exact triangles:

\[
\begin{array}{ccccccc}
A & \xrightarrow{\psi} & B & \xrightarrow{\zeta} & C & \xrightarrow{\alpha[1]} & A' \\
\downarrow{\alpha'} & \downarrow{\exists \delta'} & \downarrow{\exists \delta} & \downarrow{\id_c} & \downarrow{\alpha[1]} & \downarrow{\id_c} & \downarrow{\alpha'[1]} \\
A' & \xrightarrow{\lambda} & D & \xrightarrow{\vartheta} & C & \xrightarrow{\exists \delta'} & A'[1] \\
\end{array}
\]

and as \( \vartheta = \alpha[1] \circ \zeta \) it follows that a \( \delta \) exists that makes the diagram commutative. Now, if \( \alpha \) admits a left inverse \( \alpha' : A' \to A \), then \( \alpha'[1] \circ \vartheta = \alpha'[1] \circ \alpha[1] \circ \zeta = \zeta = \zeta \circ \id_c \), and hence \( \delta \) admits a left inverse, \( \delta' : D \to B \) and clearly \( \psi \circ \delta' = \lambda \). The converse is even simpler: if \( \psi \circ \delta' = \lambda \), then \( \alpha' \) exists and it must be a left inverse. Finally, it is obvious from the diagram that \( \alpha \) is an isomorphism if and only if \( \delta \) is. \( \square \)

We are now ready to prove our DB criterion.

**Proof of (1.5).** First observe that we may assume that \( W \) is a proper subscheme of \( X \), i.e., that \( W \neq X \). Next consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
\mathcal{I}_{W \subseteq X} & \xrightarrow{\psi} & \mathcal{O}_X & \xrightarrow{\vartheta} & \mathcal{O}_W & \xrightarrow{+1} \\
\downarrow{\vartheta'} & \downarrow{\mu} & \downarrow{\nu} & \downarrow{\nu'} & \downarrow{\mu'} & \downarrow{\nu} \\
Rf_*\mathcal{I}_{F \subseteq Y} & \xrightarrow{Rf_*\mathcal{O}_Y} & Rf_*\mathcal{O}_Y & \xrightarrow{Rf_*\mathcal{O}_F} & Rf_*\mathcal{O}_F & \xrightarrow{+1} \\
\end{array}
\]

It follows by Lemma 2.1 that there exists an object \( Q \), an exact triangle in the derived category of \( \mathcal{O}_X \)-modules,

\[
(2.1.5) \quad Q \rightarrow Rf_*\mathcal{O}_Y \oplus \mathcal{O}_W \rightarrow Rf_*\mathcal{O}_F \rightarrow +1,
\]

and a natural map \( \vartheta : \mathcal{O}_X \rightarrow Q \) that admits a left inverse, \( \vartheta' : Q \rightarrow \mathcal{O}_X \).

Now consider a similar commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
J & \xrightarrow{\Omega^0_X} & \Omega^0_W & \xrightarrow{+1} \\
\downarrow{\psi} & \downarrow{\mu} & \downarrow{\nu} & \downarrow{\nu'} & \downarrow{\mu'} & \downarrow{\nu} \\
Rf_*K & \xrightarrow{Rf_*\Omega^0_Y} & Rf_*\Omega^0_Y & \xrightarrow{Rf_*\mathcal{O}_F} & Rf_*\mathcal{O}_F & \xrightarrow{+1} \\
\end{array}
\]

Here \( J \) and \( K \) represent the appropriate objects in the appropriate derived categories that make the rows exact triangles. The vertical maps \( \mu \) and \( \nu \) exist and form a commutative square because of the basic properties of the Du Bois complex and their existence and compatibility imply the existence of the map \( \psi \) by the basic properties of derived categories.

It follows, again, by Lemma 2.1 that there exists an object \( D \), an exact triangle in the derived category of \( \mathcal{O}_X \)-modules,

\[
(2.1.6) \quad D \rightarrow Rf_*\Omega^0_Y \oplus \mathcal{O}_W \rightarrow Rf_*\Omega^0_F \rightarrow +1,
\]

and a natural map \( \delta : \Omega^0_X \rightarrow D \).
The proof of Lemma 2.1 shows that the natural transformation \( \Xi : \mathcal{O} \to \Omega^0 \) induces compatible maps between the exact triangles of (2.1.5) and (2.1.6):

\[
\begin{array}{cccc}
Q & \xrightarrow{\xi} & R\pi_*\mathcal{O}_Y \oplus \mathcal{O}_W & \xrightarrow{\eta} \xrightarrow{\zeta} R\pi_*\mathcal{O}_F \\
& & +1 & \end{array}
\]

Next, observe that if \( Y, F, \) and \( W \) are all DB, then \( \eta \) and \( \zeta \) are natural isomorphisms. Then it follows that \( \xi \) is a natural isomorphism as well. Therefore, we obtain a natural map,

\[
u' = \vartheta' \circ \xi^{-1} \circ \delta : \Omega^0_X \to \mathcal{O}_X.
\]

We want to prove that \( \nu' \) is a left inverse to the natural map \( \nu : \mathcal{O}_X \to \Omega^0_X \), i.e., that \( \nu' \circ \nu = \text{id}_{\mathcal{O}_X} \). To see this, restrict all the maps to a non-empty open set \( U \subseteq X \) such that \( U \) is smooth and \( U \cap W = \emptyset \). Such a \( U \) exists since we assumed that \( W \neq X \). It follows from the construction that

\[
\mathcal{O}_U \simeq \mathcal{O}_X|_U \xrightarrow{\vartheta'|_U} Q|_U \simeq Rf_*\mathcal{O}_Y|_U
\]

is the “usual” natural map and similarly

\[
\mathcal{O}_U \xrightarrow{\vartheta'|_U} \Omega^0_X|_U \xrightarrow{\delta|_U} D|_U \simeq Rf_*\Omega^0_Y|_U \xrightarrow{\vartheta'|_U} Rf_*\mathcal{O}_Y|_U
\]

is the same “usual” natural map. Therefore \( \nu' \circ \nu|_U = \vartheta' \circ \vartheta|_U = \text{id}_{\mathcal{O}_U} \). However, as \( X \) is reduced this implies that then \( \nu' \circ \nu = \text{id}_{\mathcal{O}_X} \) as required. Then the statement follows from [Kov99, Thm. 2.3].

We have a similar statement for seminormality. The proof is however much more elementary.

**Proposition 2.2.** Let \( f : Y \to X \) be a proper morphism between reduced schemes of finite type over \( \mathbb{C} \), \( W \subseteq X \) an arbitrary subscheme, and \( F := f^{-1}(W) \), equipped with the induced reduced subscheme structure. Assume that the natural map \( \mathcal{I}_{W \subseteq X} \to f_*\mathcal{I}_{F \subseteq Y} \) is an isomorphism. Then if \( Y \) and \( W \) are seminormal, then so is \( X \).

**Proof.** Let \( \mathcal{O}_W^\text{sn} \) be the seminormalization of \( \mathcal{O}_W \) in \( f_*\mathcal{O}_F \) and \( \mathcal{O}_X^\text{sn} \) the seminormalization of \( \mathcal{O}_X \) in \( f_*\mathcal{O}_Y \). It follows from the assumption \( \mathcal{I}_{W \subseteq X} \simeq f_*\mathcal{I}_{F \subseteq Y} \) that \( \mathcal{O}_W^\text{sn} \simeq \mathcal{O}_X^\text{sn} \). Now if \( W \) is seminormal (in fact it is enough if \( \mathcal{O}_W \) is seminormal in \( f_*\mathcal{O}_F \)), then this implies that \( \mathcal{O}_X/\mathcal{I}_{W \subseteq X} \simeq \mathcal{O}_W/\mathcal{I}_{W \subseteq X} \simeq \mathcal{O}_X^\text{sn}/\mathcal{I}_{W \subseteq X} \) and hence \( \mathcal{O}_X \simeq \mathcal{O}_X^\text{sn} \). Finally if \( Y \) is seminormal, then this implies that so is \( X \).

**Corollary 2.3.** Let \( g : X' \to X \) be a finite surjective morphism between normal varieties. Let \( Z \subseteq X \) be a reduced subscheme and assume that \( Z' := g^{-1}(Z) \) is DB. Then so is \( Z \).

**Remark 2.4.** The special case of this statement when \( Z = X \) was proved in [Kol95, 12.8.2] for \( X \) projective and in [Kov99, 2.5] in general.

**Proof.** The normalized trace map of \( X' \to X \) splits \( \mathcal{O}_Z \to f_*\mathcal{O}_{Z'} \). The rest follows from (1.5) applied to \( Z \) with \( W = \emptyset \).
3. DLT MODELS AND TWISTED HIGHER DIRECT IMAGES OF DUALIZING SHEAVES

We will frequently use the following statement in order to pass from an lc pair to its dlt model. Please recall the definition of a boundary and a minimal dlt model from (1.8).

**Theorem 3.1** (Hacon). Let \((X, \Delta)\) be a pair where \(\Delta\) is a boundary. Then there exists a \(\mathbb{Q}\)-factorial minimal dlt model \(f^m : (X^m, \Delta^m) \to (X, \Delta)\).

**Proof.** Let \(f : (X', \Delta') \to (X, \Delta)\) be a log resolution that is a composite of blow-ups of centers of codimension at least 2. Write \(K_{X'} + \Delta' \sim f^*(K_X + \Delta)\). Then there exists an effective \(f\)-exceptional divisor \(C\) such that \(-C\) is \(f\)-ample. Let \(F\) denote the part of \(\Delta'\) that consists of all \(f\)-exceptional divisors with discrepancy \(> -1\), \(E^+\) the sum of all (not necessarily exceptional) divisors with discrepancy \(\leq -1\), and let \(E := \text{red } E^+\). Note that \(E^+ - E\) is \(f\)-exceptional. Let \(H\) be sufficiently ample on \(X\). Choose \(0 < \varepsilon, \nu, \mu \ll 1\) and note that

\[
(E + (1 - \nu)F + \mu(-C + f^*H) = (1 - \varepsilon \mu)E + (1 - \nu)F + \mu(\varepsilon E - C + f^*H).
\]

Here \(-C + f^*H\) and \(\varepsilon E - C + f^*H\) are both ample, hence \(\mathbb{Q}\)-linearly equivalent to divisors \(H_1\) and \(H_2\) such that \(E + F + f_{\ast}^{-1}\Delta + H_1 + H_2\) has snc support. Furthermore, let \(\Delta_{< 1} = \Delta - [\Delta]\) denote the part of \(\Delta\) with coefficient strictly less than 1. Then

\[
(X', (1 - \varepsilon \mu)E + (1 - \nu)F + f_{\ast}^{-1}\Delta_{< 1} + \mu H_2)
\]

is klt and hence by [BCHM06] it has a minimal model \(f^m : (X^m, \Delta^m_{\varepsilon, \mu, \nu}) \to X\). By (3.1.1) this is also a minimal model of the pair \((X', E + (1 - \nu)F + f_{\ast}^{-1}\Delta_{< 1} + \mu H_1)\), which is therefore dlt. Let \(\Delta^m\) denote the birational transform of \(\Delta'\) on \(X^m\). Since \(E + (1 - \nu)F + f_{\ast}^{-1}\Delta_{< 1} \geq \Delta'\), we obtain that \((X^m, \Delta^m)\) is dlt.

For any divisor \(G \subset X'\) (e.g., \(E, F, C, H_i\)) appearing above let \(G^m\) denote its birational transform on \(X^m\). By construction

\[
N := K_{X^m} + (1 - \varepsilon \mu)E^m + (1 - \nu)F^m + (f^m)^{-1}\Delta^m_{< 1} + \mu H_2^m
\]

is \(f^m\)-nef and \(T := K_{X^m} + \Delta^m\) is \(\mathbb{Q}\)-linearly \(f^m\)-trivial. Let

\[
D^m := \Delta^m - E^m - (1 - \nu)F^m - (f^m)^{-1}\Delta^m_{< 1} + \mu C^m.
\]

Then \(-D^m\) is \(\mathbb{Q}\)-linearly \(f^m\)-equivalent to the difference \(N - T\):

\[
-D^m \sim_{f^m, \mathbb{Q}} -\Delta^m + (1 - \varepsilon \mu)E^m + (1 - \nu)F^m + (f^m)^{-1}\Delta^m_{< 1} + \mu H_2^m,
\]

hence it is \(f^m\)-nef. Since \(f^m(D^m) = f^m(\Delta^m - (f^m)^{-1}\Delta) = 0\), we see that \(D^m\) is effective by [KM98 3.39]. A divisor in \(F\) appears in \(\mu C + \Delta'\) with coefficient \(< 1\) and in \((1 - \nu)F\) with coefficient \((1 - \nu)\) and so every divisor in \(F\) has a negative coefficient in \(\Delta - E - (1 - \nu)F - f_{\ast}^{-1}\Delta + \mu C\). Therefore \(F\) gets killed in \(X^m\), so every \(f^m\)-exceptional divisor has discrepancy \(\leq -1\) and hence \((X^m, \Delta^m)\) is a minimal dlt model of \((X, \Delta)\). \(\square\)

**Theorem 3.2.** Let \(X\) be a smooth variety over \(\mathbb{C}\) and \(D = \sum a_i D_i\) an effective, integral snc divisor. Let \(\mathcal{L}\) be a line bundle on \(X\) such that \(\mathcal{L}^m \simeq \mathcal{O}_X(D)\) for some \(m \geq \max\{a_i\}\). Let \(f : X \to S\) be a projective morphism. Then the sheaves \(R^i f_*(\omega_X \otimes \mathcal{L})\) are torsion-free for all \(i\) and

\[
R^i f_*(\omega_X \otimes \mathcal{L}) \simeq \bigoplus_i R^i f_*(\omega_X \otimes \mathcal{L})[-i].
\]
Proof. If $D = 0$, this is [Kol86a, 2.1] and [Kol86b, 3.1]. The general case can be reduced to this as follows. The isomorphism $\mathcal{L}^m \simeq \mathcal{O}_X(D)$ determines a degree $m$-cyclic cover $\pi : Y \to X$ with a $\mu_m$-action and $\omega_X \otimes \mathcal{L}$ is a $\mu_m$-eigensubsheaf of $\pi_*\omega_Y$. In general $Y$ has rational singularities. Let $h : Y' \to Y$ be a $\mu_m$-equivariant resolution and $g : Y' \to S$ the composition $f \circ \pi \circ h$. Then $Rh_*\omega_{Y'} \simeq \omega_Y$, thus

$$Rf_* (\pi_* \omega_Y) \simeq Rf_* R\pi_* \omega_Y \simeq Rf_* R\pi_* Rh_* \omega_{Y'} \simeq Rg_* \omega_{Y'} \simeq \sum R^i f_* (\omega_X \otimes \mathcal{L})[-i] \simeq \sum R^i f_* (\pi_* \omega_Y)[-i]$$

by [Kol86b, 3.1] and because $\pi$ is finite. As all of these isomorphisms are $\mu_m$-equivariant, taking $\mu_m$-eigensubsheaves on both sides, we obtain the desired statement. Notice that in particular we have proven that $R^i f_* (\omega_X \otimes \mathcal{L})$ is a subsheaf of $R^i g_* \omega_{Y'}$, which is torsion-free by [Kol86a, 2.1].

4. Splitting over the non-klt locus

In the following theorem we show that the DB criterion (1.5) holds in an important situation.

Theorem 4.1. Let $f : Y \to X$ be a proper morphism with connected fibers between normal varieties. Assume that $(Y, \Delta)$ is lc and $K_Y + \Delta \sim_{\mathbb{Q}, f} 0$. Set $W := f(\text{nlkt}(Y, \Delta))$ and assume that $W \neq X$. Choose $\pi : \tilde{Y} \to Y$, an embedded resolution of $f^{-1}(W)$, such that $F := \tilde{f}^{-1}(W)$ is an snc divisor, where $\tilde{f} := f \circ \pi$. Then the natural map

$$\varrho : \mathcal{A}_W \simeq \tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F)$$

has a left inverse.

Proof. First, observe that if $\tau : \tilde{Y} \to \tilde{Y}$ is a log resolution of $(\tilde{Y}, F)$ that factors through $\pi$, then it is enough to prove the statement for $\sigma = \pi \circ \tau$ instead of $\pi$. Indeed let $\tilde{F} = \tau^{-1} F$, an snc divisor, and $\tilde{f} = f \circ \sigma$. Suppose that the natural map

$$\hat{\varrho} : \mathcal{A}_W \simeq \tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F}) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F})$$

has a left inverse, $\delta : R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F}) \to \tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F})$ such that $\delta \circ \hat{\varrho} = \text{id}_{\mathcal{A}_W}$. Then, as $\tilde{f} = \tilde{f} \circ \tau$, one has that $R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F}) \simeq R\tilde{f}_* R\tau_* \mathcal{O}_{\tilde{Y}}(-\tilde{F})$ and applying the functor $R\tilde{f}_*$ to the natural map $$\mathcal{O}_{\tilde{Y}}(-F) \simeq \tau_* \mathcal{O}_{\tilde{Y}}(-\tilde{F}) \to R\tau_* \mathcal{O}_{\tilde{Y}}(-\tilde{F})$$ shows that $\hat{\varrho} = R\tilde{f}_*(\overline{\varrho}) \circ \varrho$:

$$\hat{\varrho} : \mathcal{A}_W \simeq \tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \xrightarrow{\varrho} R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \xrightarrow{R\tilde{f}_*(\overline{\varrho})} R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-\tilde{F}).$$

Therefore, $\delta = \hat{\varrho} R\tilde{f}_*(\overline{\varrho})$ is a left inverse to $\varrho$ showing that it is indeed enough to prove the statement for $\sigma$. In particular, we may replace $\pi$ with its combination with any further blow up. We will use this observation throughout the proof.

Next write

$$\pi^*(K_Y + \Delta) \sim_{\mathbb{Q}} K_{\tilde{Y}} + E + \Delta - B,$$

where $E$ is the sum of all (not necessarily exceptional) divisors with discrepancy $-1$, $B$ is an effective exceptional integral divisor, and $|\Delta| = 0$. We may assume that $\tilde{f}^{-1} f(E)$ is an snc
divisor. Since $B - E \geq -F$, we have natural maps

$$\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E).$$

Note that $B - E \sim_{Q,f} K_{\tilde{Y}} + \Delta$, hence by (3.2)

$$R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E) \cong \sum_i R^i\tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E)[-i].$$

Thus we get a morphism

$$\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) \to R\tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E) \to \tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E).$$

Note that $\pi_* \mathcal{O}_{\tilde{Y}}(B - E) = \mathcal{I}_{nkl(Y, \Delta)}$. Furthermore, for any $U \subset X$ with preimage $U_Y := f^{-1}(U)$, a global section of $\mathcal{O}_{U_Y}$ vanishes along a fiber of $f$ if and only if it vanishes at one point of that fiber. Thus

$$\tilde{f}_* \mathcal{O}_{\tilde{Y}}(-F) = f_* \mathcal{I}_{nkl(Y, \Delta)} = \tilde{f}_* \mathcal{O}_{\tilde{Y}}(B - E).$$

5. LOG CANONICAL CENTERS

We need the following higher dimensional version of a result of Shokurov [Kol92, 12.3.1].

**Proposition 5.1.** Let $f : Y \to X$ be a proper morphism with connected fibers between normal varieties. Assume that $(Y, \Delta)$ is lc and $K_Y + \Delta \sim_{Q,f} 0$. Then, for any $x \in X$, $f^{-1}(x) \cap \text{nkl}(Y, \Delta)$ is either

(5.1.1) connected, or

(5.1.2) has 2 connected components, both of which dominate $X$ and $(Y, \Delta)$ is plt near $f^{-1}(x)$.

**Proof.** We may replace $(Y, \Delta)$ by a $\mathbb{Q}$-factorial dlt model by (3.1). After restricting to an étale local neighborhood of $x \in X$ we may assume that $\text{nkl}(Y, \Delta)$ and $f^{-1}(x) \cap \text{nkl}(Y, \Delta)$ have the same number of connected components.

Write $\Delta = E + \Delta'$ where $E = \text{nkl}(Y, \Delta) = [\Delta]$ and $(Y, \Delta')$ is klt. Let $E = \sum E_i$ be the decomposition to a sum of the connected components. Pushing forward

$$0 \to \mathcal{O}_Y(-E) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$$

we obtain

$$0 \to f_* \mathcal{O}_Y(-E) \to \mathcal{O}_X \to \sum_i f_* \mathcal{O}_{E_i} \to R^1 f_* \mathcal{O}_Y(-E).$$

Note that $-E \sim_{Q,f} K_Y + \Delta'$, hence by (5.2) (cf. [Amb03, 3.2], [Fuj08, 2.54]) $R^1 f_* \mathcal{O}_Y(-E)$ is torsion free.

Suppose $E_1$ does not dominate $X$. Then $f_* \mathcal{O}_{E_1}$ is a nonzero torsion sheaf, hence the induced map $f_* \mathcal{O}_{E_1} \to R^1 f_* \mathcal{O}_Y(-E)$ must be zero. This implies that $f_* \mathcal{O}_{E_1} \subseteq \text{im} \{ \mathcal{O}_X \to \sum_i f_* \mathcal{O}_{E_i} \}$. Further observe that the natural map $\sum_i f_* \mathcal{O}_{E_i} \to f_* \mathcal{O}_{E_1}$ gives a splitting of this embedding. On the other hand, $\text{im} \{ \mathcal{O}_X \to \sum_i f_* \mathcal{O}_{E_i} \}$ has only one generator locally near $x$, hence $f_* \mathcal{O}_{E_1} = \text{im} \{ \mathcal{O}_X \to \sum_i f_* \mathcal{O}_{E_i} \}$. In particular, there is at most one $E_i$ that does not dominate $X$. Furthermore, if $E_j$ does dominate $X$ then $\mathcal{O}_X \to f_* \mathcal{O}_{E_j}$ is nonzero. This again would contradict $f_* \mathcal{O}_{E_1} = \text{im} \{ \mathcal{O}_X \to \sum_i f_* \mathcal{O}_{E_i} \}$. Therefore, if $E$ has more than one component, then they all dominate $X$. 

LOG CANONICAL SINGULARITIES ARE DU BOIS
Until now the statement and the proof could have been done birationally, but for the rest we use the MMP repeatedly. Note that the proof is a bit messier than [Kol92, 12.3.1] since we do not have the full termination of MMP.

First we run the $(Y, (1 - \varepsilon)E + \Delta')$-MMP cf. [BCHM06, 1.3.2]. Every step is numerically $K_Y + \Delta$-trivial, hence, by the usual connectedness (cf. [KM98, 5.48]) the $E_i$ stay disjoint. At some point, we must encounter a Fano-contraction $\gamma : (Y^*, (1 - \varepsilon)E^* + \Delta^*) \rightarrow S$ where $E^*$ is ample on the general fiber. As we established above, every connected component of $E^*$ dominates $S$. We may assume that $E^*$ is disconnected as otherwise we are done.

Since the relative Picard number is one, every connected component of $E^*$ is relatively ample. As $E^*$ is disconnected, all fibers are 1-dimensional. As $\gamma$ is a Fano-contraction, the generic fiber is $\mathbb{P}^1$ and so $E^*$ can have at most, and hence exactly, two connected components, $E_1^*$ and $E_2^*$, which are sections due to the fact that the relative dimension of $\gamma$ is 1. It follows that every fiber is irreducible and so outside a codimension 2 set on the base, $\gamma : Y^* \rightarrow S$ is a $\mathbb{P}^1$-bundle with two disjoint sections. It also follows that $\Delta^*$ does not intersect the general fiber, hence $\Delta^* = \gamma^* \Delta_S$ for some $\Delta_S \subset S$ and then since the $E_i^*$ are sections we have that $(E_i^*, \Delta^*)|_{E_i^*} = (S, \Delta_S)$.

We need to prove that $(Y^*, E_1^* + E_2^* + \Delta^*)$ is plt and for that it is enough to show that $(E_i^*, \Delta^*)|_{E_i^*}$ is klt for $i = 1, 2$. By the above observation, all we need to prove then is that $(S, \Delta_S)$ is klt. Since $\gamma$ is a $\mathbb{P}^1$-bundle (in codimension 1) with 2 disjoint sections, we have that $K_{Y^*} + E_1^* + E_2^* \sim \gamma^* K_S$ and then that $K_{Y^*} + E_1^* + E_2^* + \Delta^* \sim_{Q} \gamma^* (K_S + \Delta_S)$. Now we may apply [Kol92, 20.3.3] to a general section of $Y^*$ mapping to $S$ to get that $(S, \Delta_S)$ is klt. \qed

We are now ready to prove our main connectivity theorem.

**Proof of (1.6).** We may replace $(Y, \Delta)$ by a $\mathbb{Q}$-factorial dlt model by (3.1). Hence we may assume to start with that $(Y, \Delta)$ is dlt and $Z_1, Z_2 \subset |\Delta|$ are divisors. We may also assume that $Z_2$ is disjoint from the generic fiber of $f$ and then localize at a generic point of $f(Z_1) \cap f(Z_2)$ and reduce to the case when $x := f(Z_1) \cap f(Z_2)$ is closed point.

If $f^{-1}(x) \cap |\Delta|$ is disconnected, then by (5.1), both of the $Z_i$ dominate $X$. Thus $X$ is a point and we are done. Otherwise, $f^{-1}(x) \cap |\Delta|$ is connected, and there are irreducible divisors

$$V_1 := Z_2, V_2, \ldots, V_m := Z_1 \quad \text{with} \quad V_i \subset |\Delta|$$

such that $f^{-1}(x) \cap V_i \cap V_{i+1} \neq \emptyset$ for $i = 1, \ldots, m - 1$. By working in the étale topology on $X$, we may also assume that each $f^{-1}(x) \cap V_i$ is connected.

Next, we prove by induction on $i$ that every irreducible component of

$$W_i := V_i \cap \bigcap_{j < i} f^{-1}(f(V_j))$$

is an lc center of $(Y, \Delta)$.

For $i = 1$ the statement of (5.1.1) simply says that $V_1 = Z_2$ is an lc center of $(Y, \Delta)$. Next we go from $i$ to $i + 1$. Consider $(V_i, \text{Diff}_{V_i}(\Delta - V_i))$. Note that by induction every irreducible component of $W_i$ is an lc center of $(V_i, \text{Diff}_{V_i}(\Delta - V_i))$ and $V_i \cap V_{i+1}$ is also an lc center of $(V_i, \text{Diff}_{V_i}(\Delta - V_i))$. Thus, by induction on the dimension, $(Y \leftarrow V_i, Z_1 \leftarrow V_i \cap V_{i+1})$ we conclude that every irreducible component of $f^{-1}(f(W_i)) \cap V_i \cap V_{i+1}$ is an lc center of

$$(V_i, \text{Diff}_{V_i}(\Delta - V_i)).$$
By adjunction, this shows that every irreducible component of \( f^{-1}(f(W_i)) \cap V_i \cap V_{i+1} \) is an lc center of
\[
(V_i \cap V_{i+1}, \text{Diff}_{V_i \cap V_{i+1}}(\Delta - V_i - V_{i+1})).
\]
Now looking at \( V_i \cap V_{i+1} \subset V_{i+1} \) and using inversion of adjunction, we see that every irreducible component of \( f^{-1}(f(W_i)) \cap V_i \cap V_{i+1} \) is an lc center of
\[
(V_{i+1}, \text{Diff}_{V_{i+1}}(\Delta - V_{i+1})).
\]
Using induction on the dimension again on \( f : (V_{i+1}, \text{Diff}_{V_{i+1}}(\Delta - V_{i+1})) \to X \), we see that every irreducible component of \( W_{i+1} \) is an lc center of \((Y, \Delta)\). At the end we obtain that every irreducible component of \((5.1.2)\)
\[
W_m = Z_1 \cap f^{-1}(f(Z_2)) \cap f^{-1}(f(V_2)) \cap \cdots \cap f^{-1}(f(V_{m-1}))
\]
is an lc center of \((Y, \Delta)\). \( W_m \) contains \( Z_1 \cap f^{-1}(x) \) and is contained in \( Z_1 \cap f^{-1}(f(Z_2)) \). These two are the same, hence we are done. \( \square \)

**Remark 5.1.3.** The statement of \((1.6)\) is considerably stronger than that has been previously known \[Kaw97, 1.5\], \[Amb03, 4.8\], \[Fuj08, 3.45\]. The usual claim in a similar situation has been that every irreducible component of \( f(Z_1) \cap f(Z_2) \) is dominated by an lc center.

It would also be interesting to find a proof of \((1.6)\) without using the MMP.

**Definition 5.2.** Let \( X \) be a normal scheme. A *minimal quasi log canonical structure* or simply a *minimal qlc structure* on \( X \) is a proper surjective morphism \( f : (Y, \Delta) \to X \) where
\begin{align*}
(5.2.1) & \quad (Y, \Delta) \text{ is a log canonical pair}, \\
(5.2.2) & \quad \Delta \text{ is effective}, \\
(5.2.3) & \quad \mathcal{O}_X \cong f_* \mathcal{O}_Y, \text{ and} \\
(5.2.4) & \quad K_Y + \Delta \sim_{f, \mathbb{Q}} 0.
\end{align*}

**Remark 5.2.5.** This definition is similar to Ambro’s and Fujino’s definition of a quasi-log variety \[Amb03, 4.1\], \[Fuj08, 3.29\]. The main difference here, underscored by the word “minimal” in the definition, is the additional assumption \((5.2.4)\).

One should also note that what Fujino calls a quasi-log variety is essentially \( X \) together with a qlc stratification which we define next.

**Definition 5.3.** Let \( X \) be a normal scheme and assume that it admits a minimal qlc structure \( f : (Y, \Delta) \to X \). We define the *qlc stratification of \( X \) with respect to \( f \)* or simply the *\( f \)-qlc stratification* the following way: Let \( \mathcal{H}_Y \) denote the set containing all the lc centers of \((Y, \Delta)\) and \( Y \) itself. For each \( Z \in \mathcal{H}_Y \) let
\[
W_Z := f(Z) \setminus \bigcup_{Z' \in \mathcal{H}_Y} f(Z').
\]
Further let
\[
\mathcal{H}_{X,f} = \{ W_Z | Z \in \mathcal{H}_Y \}.
\]
Notice that it is possible that \( W_Z = W_{Z'} \) for some \( Z \neq Z' \), but in \( \mathcal{H}_{X,f} \) they are only counted once. Then
\[
X = \coprod_{W \in \mathcal{H}_{X,f}} W
\]
will be called the \textit{qlc stratification of $X$ with respect to $f$} and the strata the \textit{$f$-qlc strata}.

**Definition 5.4.** Let $X_i$ be varieties that admit minimal qlc structures, $f_i : (Y_i, \Delta_i) \to X_i$ and let $W_i = \bigcup_{j=1}^{n_i} W_{i,j}$ be unions of some $f$-qlc strata on $X_i$ for $i = 1, 2$. Assume that there exists a morphism $\alpha : W_1 \to W_2$. Then we will say that $\alpha$ is a \textit{qlc stratified morphism} if for every $f$-qlc stratum $W_{1,j}$ there exists an $f$-qlc stratum $W_{2,j}$ such that $\alpha(W_{1,j}) = W_{2,j}$.

Using our new terminology we have the following important consequence of (1.6).

**Corollary 5.5.** Let $X$ be a normal variety that admits a minimal qlc structure, $f : (Y, \Delta) \to X$. Then the closure of any union of some $f$-qlc strata is also a union of some $f$-qlc strata.

In (1.10) we observed that DB singularities are seminormal, so it follows from Theorem 6.2 that the closure of any union of $f$-qlc strata is seminormal. On the other hand it also follows from the somewhat simpler [5.5] and similar results from [Fuj08].

**Proposition 5.6** [Amb03, Fuj08 §3]. Let $X$ be a normal variety that admits a minimal qlc structure, $f : (Y, \Delta) \to X$. Then each $f$-qlc stratum is normal and the closure of any union of $f$-qlc strata is seminormal.

**Proof.** Let $T$ be the closure of a union of some $f$-qlc strata. Then by Corollary 5.5 and [Fuj08 3.39(i)] (cf. [Amb03 4.4]) the qlc centers of $T$ are exactly the $f$-qlc strata (of $X$) that lie inside $T$. It follows by [Fuj08 3.33] that $T$ is seminormal and by [Fuj08 3.44] (cf. [Amb03 4.7]) that each $f$-qlc stratum is normal.

**Corollary 5.7.** Let $X$ be a normal variety that admits a minimal qlc structure, $f : (Y, \Delta) \to X$. Then the support of the conductor subscheme of the closure of any union of $f$-qlc strata is contained in a smaller dimensional union of $f$-qlc strata.

**Proof.** As individual $f$-qlc strata are normal, it follows that the conductor subscheme is contained in the part of the closure that was subtracted in (5.3). By (1.6) this is a union of $f$-qlc strata and as it does not contain any (maximal) component of the original union, the dimension of each contributing strata has to be strictly smaller.

### 6. Log Canonical Singularities are Du Bois

**Lemma 6.1.** Let $X$ be a normal variety and $f : (Y, \Delta) \to X$ a minimal qlc structure on $X$. Let $W \in H_{X,f}$ be a qlc stratum of $X$ and $\overline{W}$ its closure in $X$. Then there exist a normal variety $\overline{W}$ with a minimal qlc structure $g : (Z, \Sigma) \to \overline{W}$ such that $g(\text{nklt}(Z, \Sigma)) \neq \overline{W}$ and a finite surjective qlc stratified morphism $\overline{W} \to \overline{W}$.

**Proof.** note that we may replace $(Y, \Delta)$ by a $\mathbb{Q}$-factorial dlt model cf. (3.1). Recall that then $\overline{W}$ is the union of some $f$-qlc strata by (5.5). If $\overline{W} = X$ and $g(\text{nklt}(Z, \Sigma)) \neq X = \overline{W}$ then choosing $(Z, \Sigma) = (Y, \Delta)$, $g = f$, and $\overline{W} = X$ the desired conditions are satisfied. Otherwise, there exists an irreducible component $E \subseteq [\Delta]$ such that $\overline{W} \subseteq f(E)$. Consider the Stein factorization of $f|_E$:

$$f|_E : E \xrightarrow{f_E} G \xrightarrow{\sigma} f(E).$$

Observe that then $f_E : (E, \text{Diff}_E \Delta) \to G$ is a minimal qlc structure, $G$ is normal, and $\sigma$ is finite. Let $W_1 = \sigma^{-1}(W)$ denote the preimage of $W$, and $\overline{W}_1$ its closure in $G$. By (1.6) the $f_E$-qlc stratification of $G$ is just the preimage of the restriction of the $f$-qlc stratification
of $X$ to $f(E)$, so the induced morphism $\overline{W}_1 \rightarrow \overline{W}$ is a qlc stratified morphism and as long as $\overline{W} \neq f(E)$ or $f(\text{nklt}(E, \text{Diff}_E \Delta)) = f(E)$ we may repeat this step with $X$ replaced with $G$ and $W$ replaced with $\sigma^{-1}(W)$ without changing the induced qlc structure on $W$. By noetherian induction this process must end and at one point we will have $\overline{W} = f(E)$ and $f(\text{nklt}(E, \text{Diff}_E \Delta)) \neq f(E)$. Then $f_E : (E, \text{Diff}_E \Delta) \rightarrow G$ and $\sigma : \widehat{W} := G \rightarrow \overline{W}$ satisfy the desired conditions.

□

Theorem 1.4 is implied by the following.

**Theorem 6.2.** If $X$ admits a minimal qlc structure, $f : (Y, \Delta) \rightarrow X$, then the closure of any union of $f$-qlc strata is DB.

**Proof.** Let $T \subseteq X$ be a union of $f$-qlc strata. By (5.5) we know that $\overline{T}$, the closure of $T$ in $X$, is also a union of $f$-qlc strata, so by replacing $T$ with $\overline{T}$ we may assume that $T$ is closed. Let $\overline{T}$ denote the normalization of $T$. We have that $T = \bigcup_{W \in \mathcal{J}} W$ for some $\mathcal{J} \subseteq \mathcal{H}_{X,f}$, so $T$ is seminormal by (5.6). For $W \in \mathcal{J}$, we will denote the closure of $W$ in $X$ by $\overline{W}$. Note that by definition $\overline{W}$ is contained in $T$. In order to prove that $T$ is DB, we will apply a double induction the following way:

- **induction on $\dim X$**: Assume that the statement holds if $X$ is replaced with a smaller dimensional variety admitting a minimal qlc structure.
- **induction on $\dim T$**: Assume that the statement holds if $X$ is fixed and $T$ is replaced with a smaller dimensional subvariety of $X$ which is also a union of $f$-qlc strata.

First assume that $X \neq T$. Then $\overline{W}$ must also be a proper subvariety of $X$ for any $W \in \mathcal{J}$. Then by (6.1) for each $W \in \mathcal{J}$ there exists a normal variety $\widehat{W}$ with a minimal qlc structure and a finite surjective qlc stratified morphism $\sigma : \widehat{W} \rightarrow \overline{W}$. By induction on $\dim X$ we obtain that $\widehat{W}$ is DB. Then by (2.3) it follows that the normalization of $\overline{W}$ is DB as well. Note that $\widehat{W}$ is normal, but may not be the normalization of $\overline{W}$, however $\sigma$ factors through the normalization morphism.

Let $\mathcal{J}' \subseteq \mathcal{J}$ be a subset such that $T = \bigcup_{W \in \mathcal{J}'} \overline{W}$ and $\overline{W} \not\subseteq \overline{W'}$ for any $W, W' \in \mathcal{J}'$. Then let $\widehat{T} := \coprod_{W \in \mathcal{J}'} \widehat{W}$ and $\widehat{\pi} : \widehat{T} \rightarrow T$ the natural morphism. Observe that as the $\widehat{W}$ have DB singularities, so does $\widehat{T}$ and then by (2.3) it follows that for the normalization of $\overline{T}$, $\tau : \overline{T} \rightarrow T$, $\overline{T}$ is DB as well. Next let $Z \subset T$ be the conductor subscheme of $T$ and $\overline{Z}$ its preimage in $\overline{T}$. Then since $T$ is seminormal, both $Z$ and $\overline{Z}$ are reduced and

\[(6.2.1) \quad \mathcal{I}_{Z \subseteq T} = \tau_\ast \mathcal{I}_{\overline{Z} \subseteq \overline{T}}.\]

By (5.7) $Z$ is a smaller dimensional union of $f$-qlc strata and thus it is DB by induction on $\dim T$. Next let $\overline{Z} = (\tau^{-1}Z)_{\text{red}} \subseteq \overline{T}$ be the reduced preimage of $Z$ (as well as of $\overline{Z}$) in $\overline{T}$. The following diagram shows the connections between the various objects we have defined so far:
We have seen above that each $\hat{W}$ admits a minimal qlc structure compatible with the minimal qlc structure on $T$ and then by \((5.7)\) again $\hat{Z}$ is also a union of qlc strata on $\hat{T}$ and the morphism $\hat{Z} \to \hat{\Delta}$ is a qlc stratified morphism. Then since $\dim \hat{T} < \dim X$, by replacing $X$ with $\hat{T}$ shows that $\hat{Z}$ is DB by induction on $\dim X$. In turn this implies that $\hat{Z}$ is DB by \((2.3)\).

Therefore, by now we have proved that $\hat{T}$, $Z$, and $\hat{Z}$ all have DB singularities, so by using \((6.2.1)\) and \((1.5)\) we conclude that $T$ is DB as well.

Now assume that $X = T$ and hence $X = T = \hat{T}$. Let $f : (Y, \Delta) \to X$ be a minimal qlc structure and $W = f(\text{nklt}(Y, \Delta))$. By \((6.1)\) we may assume that $W \neq X$ by replacing $X$ by a finite cover. Note that by \((2.3)\) it is enough to prove that this finite cover is DB.

Then let $\pi : \hat{Y} \to Y$ be a log resolution and $F := (f \circ \pi)^{-1}(W)$, an snc divisor. By \((4.1)\) the natural map $\phi : \mathcal{I}_W = f_* \mathcal{O}_{\hat{Y}}(-F) \to Rf_* \mathcal{O}_{\hat{Y}}(-F)$ has a left inverse. Finally, then \((1.5)\) implies that $T = X$ is DB. \(\square\)

**Definition 6.3.** Let $\phi : X \to B$ be a flat morphism. We say that $\phi$ is a **DB family** if $X_b$ is DB for all $b \in B$.

**Definition 6.4.** Let $\phi : X \to B$ be a flat morphism. We say that $\phi$ is a **family with potentially lc fibers** if for all closed points $b \in B$ there exists an effective $\mathbb{Q}$-divisor $D_b \subset X_b$ such that $(X_b, D_b)$ is log canonical.

**Definition 6.5** \([\text{KM98}, 7.1]\). Let $X$ be a normal variety, $D \subset X$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier, and $\phi : X \to B$ a non-constant morphism to a smooth curve $B$. We say that $\phi$ is a **log canonical morphism** or an **lc morphism** if $(X, D + X_b)$ is lc for all closed points $b \in B$.

**Remark 6.6.** Notice that for a family with potentially lc fibers it is not required that the divisors $D_b$ also form a family over $B$. On the other hand, if $\phi : X \to B$ is a family with lc fibers, $B$ is a smooth curve and there exists an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier and $D |_{X_b} = D_b$ then $\phi$ is an lc morphism by inversion of adjunction \([\text{Kaw07}]\).

Further observe that if $\phi : (X, D) \to B$ is an lc morphism, then for any $b \in B$, choosing $(Y, \Delta) = (X, D + X_b)$ and $f : (Y, \Delta) \to X$ the identity of $X$ gives an $f$-qlc stratification of $X$ such that $X_b$ is a union of $f$-qlc strata. In particular, it follows by \((6.2)\) that $X_b$ is DB. Note that if $X_b$ is reducible, then \((1.4)\) would not suffice here.

**Corollary 6.7.** Let $\phi : X \to B$ be either a family with potentially lc fibers or an lc morphism. Then $\phi$ is a DB family.

**Proof.** Follows directly from \((6.2)\). \(\square\)
7. INVARIANCE OF COHOMOLOGY FOR DB MORPHISMS

The following notation will be used throughout this section.

**Notation 7.1.** Let $\pi : \mathbb{P}^N_B \to B$ be a projective $N$-space over $B$, $\iota : X \hookrightarrow \mathbb{P}^N_B$ a closed embedding, and $\phi := \pi \circ \iota$. Further let $\mathcal{O}_X(1)$ be a relatively ample line bundle, and $\mathcal{L} = \mathcal{O}_X(1)|_X$.

Then $\omega^*_B$ will denote the relative dualizing complex and $h^{-i}(\omega^*_B)$ its $-i$th cohomology sheaf. We will also use the notation $\omega^*_\phi := h^{-n}(\omega^*_B)$ where $n = \dim(X/B)$. Naturally these definitions automatically apply for $\pi$ in place of $\phi$ by choosing $\iota = \text{id}_{\mathbb{P}^N_B}$.

**Lemma 7.2.** Let $b \in B$. Then

$$h^{-i}(\omega^*_\phi) \simeq \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi) \quad \text{and} \quad h^{-i}(\omega^*_X) \simeq \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega^*_B).$$

In particular, $h^{-i}(\omega^*_\phi) = 0$ and $h^{-i}(\omega^*_X) = 0$ if $i < 0$ or $i > N$.

**Proof.** By Grothendieck duality ([Har66, VII.3.3], cf. [Har77, III.7.5]),

$$h^{-i}(\omega^*_\phi) \simeq h^{-i}(R\mathcal{H}om_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi)) \simeq h^{-i}(R\mathcal{H}om_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi)[N]) \simeq \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi).$$

The same argument obviously implies the equivalent statement for $h^{-i}(\omega^*_X)$.

Furthermore, clearly $\mathcal{E}xt^j_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi) = 0$ and $\mathcal{E}xt^j_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega^*_B) = 0$ if $j < 0$, and hence $h^{-i}(\omega^*_\phi) = 0$ and $h^{-i}(\omega^*_X) = 0$ if $i > N$. Since $\mathbb{P}^N_B$ is smooth and thus all the local rings are regular, it also follows that $\mathcal{E}xt^j_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega^*_B) = 0$ if $j > N$, and hence $h^{-i}(\omega^*_X) = 0$ if $i < 0$.

Next, consider the restriction map $[AK80, 1.8],

$$\rho^{-i}_b : \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega_\pi)|_{X_b} \to \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{O}_X, \omega^*_B).$$

We have just observed that the target of the map is 0 if $i < 0$. In particular, $\rho^{-i}_b$ is surjective in that range. Then by $[AK80, 1.9]$ $\rho^{-i}_b$ is an isomorphism and therefore $h^{-i}(\omega^*_\phi) = 0$ if $i < 0$. □

**Lemma 7.3.** Let $\mathcal{F}$ be a coherent sheaf on $X$ and assume that $R^i\pi_*((\mathcal{F})(-q))$ is locally free for $q \gg 0$. Then

$$\pi_*\mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{F}, (\omega_\pi)(q)) \simeq \mathcal{H}om_B(R^i\pi_*((\mathcal{F})(-q)), \mathcal{O}_B) \quad \text{for } q \gg 0.$$ 

**Proof.** Let $q \gg 0$ and $U \subseteq B$ an affine open set such that $R^i\pi_*((\mathcal{F})(-q))|_U$ is free. Then by [Har77, III.6.7] and [Har66, II.5.2],

$$\mathcal{H}^0(\pi^{-1}(U), \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{F}, (\omega_\pi)(q))) \simeq \mathcal{E}xt^i_{\mathbb{P}^N_B}(\mathcal{F}_U(-q), (\omega_\pi)_U) \simeq \mathcal{H}om_U(R^i\pi_*\mathcal{F}(-q)|_U, \mathcal{O}_U) \simeq \mathcal{H}^0(U, \mathcal{H}om_B(R^i\pi_*\mathcal{F}(-q), \mathcal{O}_B)).$$ □

We will also need the base-change theorem of Du Bois and Jarraud [DJ74, Théorème] (cf. [DB81, 4.6]):

**Theorem 7.4.** Let $\phi : X \to B$ be a projective DB family. Then $R^i\phi_*\mathcal{O}_X$ is locally free of finite rank and compatible with arbitrary base change for all $i$. □

**Remark 7.5.** This result naturally leads to the question whether a similar statement holds for higher direct images of the relative dualizing sheaf. I.e., are the sheaves $R^i\phi_*\omega_\phi$ locally free of finite rank and compatible with arbitrary base change for all $i$? By the next theorem $\omega_\phi$ is flat, but it is not obvious that its cohomologies stay constant.
The next theorem is our main flatness and base change result.

**Theorem 7.6.** Let \( \phi : X \to B \) be a projective DB family. Then

1. (7.6.1) the sheaves \( h^{-i}(\omega_X^*) \) are flat over \( B \) for all \( i \),
2. (7.6.2) the sheaves \( \phi_* (h^{-i}(\omega^*_\phi) \otimes \mathcal{L}^q) \) are locally free and compatible with arbitrary base change for all \( i \) and for all \( q \gg 0 \), and
3. (7.6.3) for any base change morphism \( \vartheta : T \to B \) and for all \( i \),

\[
(h^{-i}(\omega_X^*))_T \simeq h^{-i}(\omega^*_{\vartheta_T}).
\]

**Remark 7.7.** For a coherent sheaf \( \mathcal{F} \) on \( X \), the pushforward \( \phi_* \mathcal{F} \) being compatible with arbitrary base change means that for any morphism \( \vartheta : T \to B \),

\[
(\phi_* \mathcal{F})_T \simeq (\vartheta)_* (\phi_T)_* \mathcal{F}_T.
\]

In particular, (7.6.2) implies that for any \( \vartheta : T \to B \),

\[
(\phi_* (h^{-i}(\omega^*_\phi) \otimes \mathcal{L}^q))_T \simeq (\vartheta)_* (h^{-i}(\omega^*_{\vartheta_T}) \otimes \mathcal{L}_T^q).
\]

Combined with (7.6.3) this means that for any \( \vartheta : T \to B \),

\[
(\phi_* (h^{-i}(\omega^*_\phi) \otimes \mathcal{L}^q))_T \simeq (\vartheta)_* (h^{-i}(\omega^*_{\vartheta_T}) \otimes \mathcal{L}_T^q).
\]

Arguably, (7.7.1) is the natural base change statement one might hope for, but it should be noted that the analogous statement does not necessarily hold on the level of the dualizing complexes.

**Proof.** We may assume that \( B = \text{Spec} R \) is affine. By definition, \( \mathcal{L}^m \) is relatively generated by global sections for all \( m \in \mathbb{N} \). For a given \( m \in \mathbb{N} \), choose a general section \( \vartheta \in H^0(X, \mathcal{L}^m) \) and consider the \( \mathcal{O}_X \)-algebra

\[
\mathcal{A}_m = \bigoplus_{j=0}^{m-1} \mathcal{L}^{-j} \simeq \bigoplus_{j=0}^{\infty} \mathcal{L}^{-j} t^j \bigg/ (t^m - \vartheta)
\]

as in [KM98, 2.50]. Let \( Y_m^* : = \text{Spec} \mathcal{A}_m \) and \( \sigma : Y_m^* \to X \) the induced finite morphism. Then

\[
R^i (\phi \circ \sigma)_* \mathcal{O}_{Y_m^*} \simeq R^i \phi_* (\sigma_* \mathcal{O}_{Y_m^*}) \simeq R^i \phi_* \mathcal{A}_m \simeq \bigoplus_{j=0}^{m-1} R^i \phi_* \mathcal{L}^{-j}
\]

for all \( i \) and all \( b \in B \). Note that by construction, this direct sum decomposition is compatible with arbitrary base change. By [Kov99, 2.4], \( \phi \circ \sigma \) is again a DB family and hence \( R^i (\phi \circ \sigma)_* \mathcal{O}_{Y_m^*} \) is locally free and compatible with arbitrary base change by (7.4). Since \( \phi \) is flat and \( \mathcal{L} \) is locally free, it follows that then \( R^i \phi_* \mathcal{L}^{-j} \) is locally free and compatible with arbitrary base change for all \( i \) and for all \( j > 0 \). Then taking \( \mathcal{F} = \mathcal{O}_X \) and applying [Har77, III.6.7], (7.2), and (7.3), we obtain that

\[
(7.7.2) \quad \phi_* (h^{-i}(\omega^*_\phi) \otimes \mathcal{L}^q) \simeq \text{Hom}_B (R^i \phi_* \mathcal{L}^{-q}, \mathcal{O}_B) \quad \text{for } q \gg 0.
\]

This proves (7.6.2) and then (7.6.1) follows easily by an argument similar to the one used to prove the equivalence of (i) and (ii) in the proof of [Har77, III.9.9].

To prove (7.6.3) we will use induction on \( i \). Notice that it follows trivially for \( i < 0 \) (and \( i > N \), but we will not use that fact) by (7.2), so the start of the induction is covered.
Consider the pull back map,
\[ \rho_T^{-i} : \mathcal{E} \chi^{N-i}_{\mathbb{P}^N_B}(\mathcal{O}_{\mathcal{X}_T}, \omega_{\mathcal{X}_T}) \rightarrow \mathcal{E} \chi^{N-i}_{\mathbb{P}^N_B}(\mathcal{O}_{\mathcal{X}_T}, \omega_{\mathcal{X}_T}). \]

By the inductive hypothesis \( \rho_T^{-j} \) is an isomorphism and \( \mathcal{E} \chi^{N-j}_{\mathbb{P}^N_B}(\mathcal{O}_{\mathcal{X}_T}, \omega_{\mathcal{X}_T}) \cong h^{-j}(\omega_{\phi}) \) is flat over \( B \) by (7.6.1). Then by [AK80] 1.9, \( \rho_T^{-(j+1)} \) is also an isomorphism. This proves (7.6.3). □

**Lemma 7.8.** Let \( X \) be a subscheme of \( \mathbb{P}^N \), \( \mathcal{F} \) a coherent sheaf on \( X \) and \( \mathcal{N} \) a fixed line bundle on \( \mathbb{P}^N \). Then \( \mathcal{F} \) is \( S_k \) at \( x \) if and only if \( \mathcal{E} \chi_{\mathbb{P}^N}(\mathcal{F}, \mathcal{N})_x = 0 \) for all \( j > N - k \).

**Proof.** Since \( \mathcal{O}_{\mathbb{P}^N, x} \) is a regular local ring,
\[ d := \text{depth}_{\mathcal{O}_{X, x}} \mathcal{F}_x = \text{depth}_{\mathcal{O}_{\mathbb{P}^N, x}} \mathcal{F}_x = N - \text{proj dim}_{\mathcal{O}_{\mathbb{P}^N, x}} \mathcal{F}_x. \]
Therefore, \( d \geq k \) if and only if \( \text{Ext}^{j}_{\mathcal{O}_{\mathbb{P}^N, x}}(\mathcal{F}_x, \mathcal{N}_x) = 0 \) for all \( j > N - k \). □

Using our results in this section we obtain a criterion for Serre’s \( S_k \) condition, analogous to [KM98, 5.72], in the relative setting.

**Theorem 7.9.** Let \( \phi : X \rightarrow B \) be a projective DB family, \( x \in X \) and \( b = \phi(x) \). Then \( X_b \) is \( S_k \) at \( x \) if and only if
\[ h^{-i}(\omega_{\phi})_x = 0 \quad \text{for } i < k. \]

**Proof.** Let \( \mathcal{F} = \mathcal{O}_{X_b} \), \( j = N - i \) and \( \mathcal{N} = \omega_{\mathbb{P}^N_b} \). Then (7.3) and (7.8) imply that \( X_b \) is \( S_k \) at \( x \) if and only if \( h^{-i}(\omega_{X_b})_x = 0 \) for \( i < k \). Then the statement follows from (7.6.3) and Nakayama’s lemma. □

The following result asserts the invariance of the \( S_k \) property in DB families:

**Theorem 7.10.** Let \( \phi : X \rightarrow B \) be a projective DB family and \( U \subseteq X \) an open subset. Assume that \( B \) is connected and the general fiber \( U_{\text{gen}} \) of \( \phi|_U \) is \( S_k \). Then all fibers \( U_b \) of \( \phi|_U \) are \( S_k \).

**Proof.** Suppose that the fiber \( U_b \) of \( \phi|_U \) is not \( S_k \). Then by (7.9) there exists an \( i < k \) such that \( h^{-i}(\omega_{\phi})_x \neq 0 \) for some \( x \in U_b \). Let \( Z \) be an irreducible component of \( \text{supp} h^{-i}(\omega_{\phi}) \) such that \( Z \cap U_b \neq \emptyset \). It follows that \( Z \cap U \) is dense in \( Z \). By (7.6.1) \( h^{-i}(\omega_{\phi})_x \) is flat over \( B \) and thus \( Z \) and then also \( Z \cap U \) dominate \( B \). However, that implies that \( Z \cap U_{\text{gen}} \neq \emptyset \) contradicting the assumption that \( U_{\text{gen}} \) is \( S_k \) and hence the proof is complete. □

As mentioned in the introduction, our main application is the following.

**Corollary 7.11.** Let \( \phi : X \rightarrow B \) be a projective family with potentially lc fibers or a projective lc morphism and \( U \subseteq X \) an open subset. Assume that \( B \) is connected and the general fiber \( U_{\text{gen}} \) of \( \phi|_U \) is \( S_k \) (resp. CM). Then all fibers \( U_b \) of \( \phi|_U \) are \( S_k \) (resp. CM).

**Proof.** Follows directly from (6.2) and (7.10). □

The following example shows that the equivalent statement does not hold in mixed characteristic.
**Example 7.12** (Schröer). Let $S$ be an ordinary Enriques surface in characteristic $2$ (see [CD89, p. 77] for the definition of ordinary). Then $S$ is liftable to characteristic $0$ by [CD89, 1.4.1]. Let $\eta : Y \to \text{Spec } R$ be a family of Enriques surfaces such that the special fiber is isomorphic to $S$ and the general fiber is an Enriques surface of characteristic $0$.

Let $\zeta : Z \to \text{Spec } R$ be the family of the projectivized cones over the members of the family $\eta$. i.e., for any $t \in \text{Spec } R$, $Z_t$ is the projectivized cone over $Y_t$. Since $K_{Y_t} \equiv 0$ for all $t \in \text{Spec } R$, we obtain that $\zeta$ is both a projective family with potentially lc fibers, and a projective lc morphism. By the choice of $\eta$, the dimension of the cohomology group $H^1(Y_b, O_{Y_b})$ jumps: it is $0$ on the general fiber, and $1$ on the special fiber. Consequently, by (7.13), the general fiber of $\zeta$ is CM, but the special fiber is not.

Recall the following CM condition used in the above example:

**Lemma 7.13.** Let $E$ be a smooth projective variety over a field of arbitrary characteristic and $Z$ the cone over $E$. Then $Z$ is CM if and only if $h^i(E, O_E(m)) = 0$ for $0 < i < \dim E$ and $m \in \mathbb{Z}$.

**Proof.** See [Kol08, Ex. 71] and [Kov99, 3.3].

The most natural statement along these lines would be if we did not have to assume the existence of the projective compactification of the family $U \to B$. This is related to the following conjecture, which is an interesting and natural problem on its own:

**Conjecture 7.14.** Let $\psi : U \to B$ be an affine, finite type lc morphism. Then there exists a base change morphism $\vartheta : T \to B$ and a projective lc morphism $\phi : X \to B_T$ such that $U_T \subseteq X$ and $\psi_T = \phi|_U$.

We expect that (7.14) should follow from an argument using MMP techniques but it might require parts that are at this time still open, such as the abundance conjecture. On the other hand, (7.14) would clearly imply the following strengthening of (7.11):

**Conjecture-Corollary 7.15.** Let $\psi : U \to B$ be a finite type lc morphism. Assume that $B$ is connected and the general fiber of $\psi$ is $S_k$ (resp. CM). Then all fibers are $S_k$ (resp. CM).

**References**


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