6 mins a page

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Termination of flips

① Intro
② local
③ local-global
④ global.

1

Defn Let X be a variety. We say a $\mathbb{R}$-Cartier divisor D is *nef* if $D\cdot C \geq 0$ for all curves C.

Let $\pi\colon X \longrightarrow U$ be a projective morphism.

$$\overline{NE}(X/U) = \{\alpha \in H_2(X,\mathbb{R}) \mid \alpha = \sum a_i[C_i],\ a_i \geq 0,\ \pi_*\alpha = 0\}.$$

(X,Δ) log pair: X normal, $\Delta = \sum a_i \Delta_i$, $a_i \geq 0$, $a_i \in \mathbb{R}$

$K_X+\Delta$ $\mathbb{R}$-Cartier.

$f\colon Y \longrightarrow X$ log resolution

$$K_Y + \tilde{\Delta} + \sum E_i = f^*(K_X+\Delta) + \sum a_i E_i$$

$a_i = a(X,\Delta,E_i) =$ log disc of E_i wrt $K_X+\Delta$.

$a = \inf_i a_i =$ log discrepancy. $\Gamma = \tilde{\Delta} + \sum E_i - \sum a_i E_i$

(X,Δ) log canonical iff $a \geq 0$

kawamata log terminal iff $a > 0$.

$\operatorname{Nonklt}(X,\Delta) =$ image of divisors of log discrepancy zero.

$= \lfloor \Gamma \rfloor$.

Intuition log discrepancy is a measure of sings. Worse sings smaller log discrepancy.

MMP

1. Start with a klt pair $K_X+\Delta$.
2. Is $K_X+\Delta$ nef? If yes then STOP. log terminal model.
3. If not there is a $(K_X+\Delta)$-extremal ray R of $\overline{NE}(X/U)$, and a contraction morphism $\phi: X \longrightarrow Z$ which contracts those curves which generate R, $-K_X$ ample $/Z$

   (a) $\dim Z < \dim X$. STOP Mori fibre space

   (b) ϕ birational. (i) ϕ contracts a divisor Go to 2.

   (ii) ϕ ~~does~~ small.

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\phi"'] & & X^+ \arrow[dl, "\phi^+"] \\ & Z & \end{tikzcd}$$

f iso in cod 1.

ϕ^+ small

$\rho(X^+/Z) = \rho(X/Z) = 1$

K_{X^+} ample.

Example S smooth projective surface.

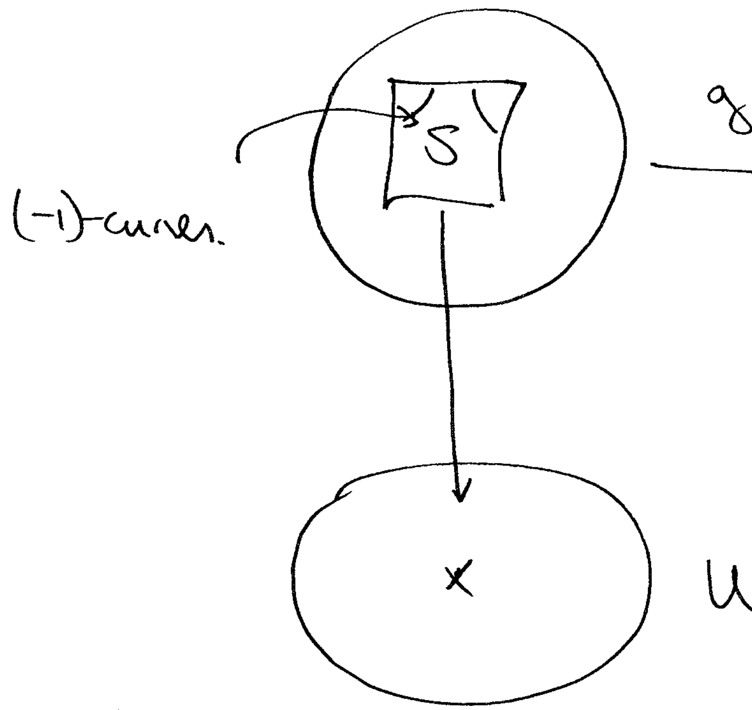
$S \longrightarrow pt$

$S \longrightarrow C$ $\mathbb{P}^1$-bundle

$S \longrightarrow T$ contract a (-1)-curve.

③ Let U = cone over S, a surface of general type.

X = blow up, a threefold $/U$



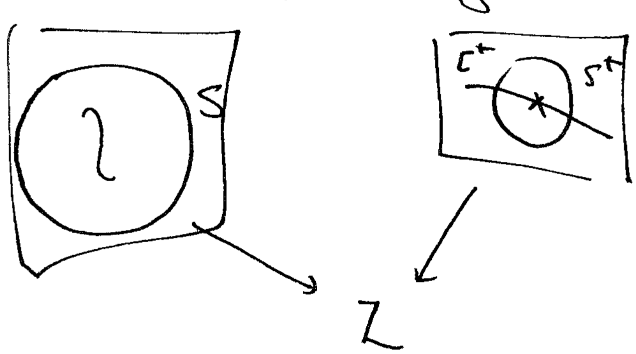
$\xrightarrow{g}$ S $\mathbb{P}^1$-bundle.

(−1)-curves in S generate extremal rays X/U.

(K_X+S) − MMP $/U$.

Cannot contract S.

We must flip every (−1)-curve in S.



$S \longrightarrow T = S^+$

contracts a (−1)-curve

<u>Thm</u> (Special Termination). Assume flips terminate in dim $n-1$. Then any sequence of flips is eventually disjoint from nonklt.

④ Basic idea.

Reduce to the case nonklt is a divisor.

Let us suppose a irreducible.

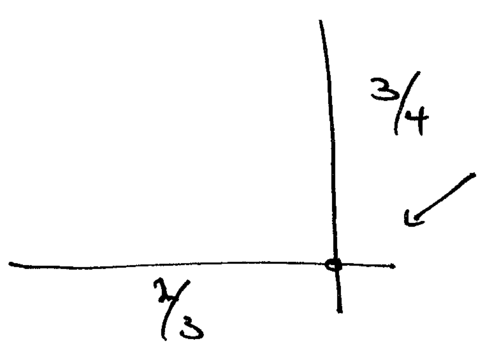
Now every flip which touches S is like a step of the $(K_S+\Theta)$-MMP.

<u>One problem</u>

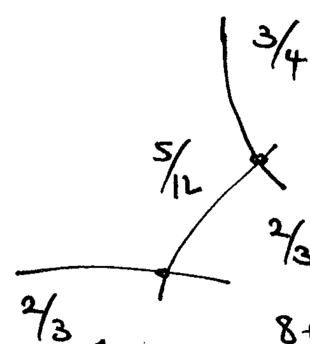
$$S \dashrightarrow S^+ \quad \text{might extract a divisor.}$$

But the coeff of this divisor must be > 0.

<u>E.g.</u>



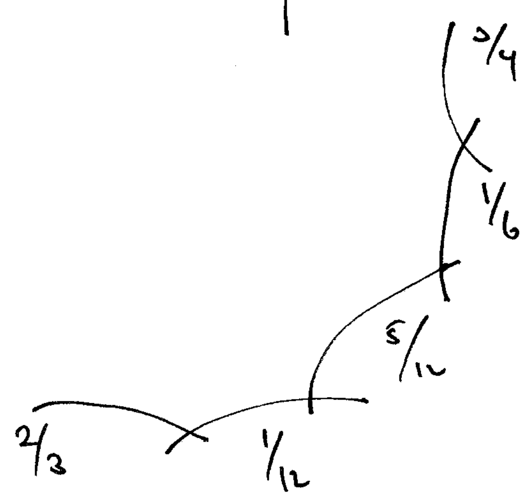
S



$$\tfrac{2}{3}+\tfrac{3}{4}-1$$

$$\frac{8+9-12}{12} = \tfrac{5}{12}$$

$$\tfrac{5}{12} < \left(\tfrac{2}{3}, \tfrac{3}{4}\right)$$



no more.

(X, Δ) klt

there are only finitely many exceptionals of log disc < 1

$$a(X,\Delta,V) = \inf_{\text{centre of } E_i = V} a_i$$

Conjecture ACC. Fix dim n, a DCC set I

$$\{\, a(X,\Delta,x) \mid \text{coeffs of } \Delta \text{ belong to } I,\ \dim X = n \,\}$$

satisfies ACC

Conjecture LSC. (X,Δ) log pair.

~~$x \in C$~~
pt. ~ curve

$\forall\, x \in X\ \exists\, U$ nbhd of x

s.t. $a(X,\Delta,x) = \inf_{y \in U} a(X,\Delta,y)$

"Log discrepancy is a measure of how bad a singularity is".

Lemma LSC $\iff$ $a(X,\Delta,V) \le a(X,\Delta,W) + \operatorname{codim}(V,W)$, $V \subset W$.

Thm (Shokurov)

Assume ACC, LSC.

Then any sequence of flips terminates.

Pf $X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots$

$\exists\, d, a$ s.t. $a = \inf \{\, V \subset X \mid \dim V = d,\ a(X,\Delta,V) = a \,\}$.

~~and the locus W_i and the locus of~~

(6)

$$W_i = \bigcup \left\{ V \subset X_i \;\middle|\; \begin{array}{l} \dim V = d \\ a(X_i, \Delta_i, V) = a \end{array} \right\}$$

then $W_i \dashrightarrow W_{i+1}$ is not an iso. in dim $\geq d$.

Since a is the min, LSC $\Longrightarrow$ W_i is closed.

$W_i \dashrightarrow W_{i+1}$ does not extract any centres ~~divisors~~ of ~~...~~ dimension d.

+ log disc of any subvariety V of dim $d = a$.

Eg $\dim X = 6$

$\dim W = 4$ $\qquad a = ?$

$d = 2$.

W is swept out by surfaces S_t.

$W_i \dashrightarrow W_{i+1}$: never introduces any new surfaces

~~But~~ And infinitely often $W_i \dashrightarrow W_{i+1}$ collapses a surface to a pt or a curve $\times$.

⑦ klt. $B \geq 0$ eff.

<u>Birkar</u>: (X, Δ) log canonical threshold
$= \sup\{t \mid (X, \Delta + tB)$ is lc$\}$.

<u>Conjecture</u> Fix n, DCC set I.

$\{t \mid \exists (X, \Delta)$ $\dim X = n$, Δ coeffs in I
B integral $t = \operatorname{lct}(X, \Delta + tB)\}$

satisfies ACC. <u>Thm</u> (de Fernex, Ein, Mustaţă)
True for X complete intersection.

<u>Thm</u> Assume MMP in dim $n-1$.

(X, Δ) projective $K_X + \Delta \underset{\mathbb{R}}{\sim} B \geq 0$.

Then any sequence of flips terminates

<u>Pf</u> Suppose $X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \dots$

is an infinite sequence of flips.

Then this is an infinite sequence of flips
for $(X, \Delta + tB)$ any $t \geq 0$.

Let $t_1 = \operatorname{lct}(X, \Delta + t_1 B)$. Caroline Araujo + Brian Lehmann

By S.T. only finitely many flips intersect
non klt. Throw away non klt. Increase t_1 to t_2
repeat. Throw away non klt. Increase t_2 to t_3 ...
~~Must stop~~ Get increasing sequence of lc thresholds #

Thm Let $\pi : X \longrightarrow U$ projective morphism, (X,Δ) l.c.
$R = \mathbb{R}^+ C$ extremal ray of $\overline{NE}(X/U)$.

$+(K_X+\Delta)\cdot C \geq -2n.$ $n = \dim X.$

Conjecture We can replace $2n$ by $n+1$.

Defn-Lemma $\pi : X \longrightarrow U$, (X,Δ) l.c
proj. morphism $\exists\, t > 0$ s.t

$\forall\ R = \mathbb{R}^+ C$ extremal ray of $\overline{NE}(X/U)$.

$$(K_X+\Delta)\cdot C > 0 \implies (K_X+\Delta)\cdot C \geq t.$$

Pf If $r(K_X+\Delta)$ is Cartier take $t = 1/r$.

In general

$$K_X+\Delta = \sum r_i (K_X+\Delta_i) \qquad K_X+\Delta_i \ \mathbb{Q}\text{-Cartier}$$

$$\sum r_i = 1, \quad r_i > 0, \quad r_i \in \mathbb{R}$$

Suppose $m_i(K_X+\Delta_i)$ Cartier.

$$(K_X+\Delta)\cdot C = \sum \frac{r_i d_i}{m_i} \qquad d_i \in \mathbb{Z}, \quad d_i \geq 2m_i n.$$

\# of these form a DCC set.

Global

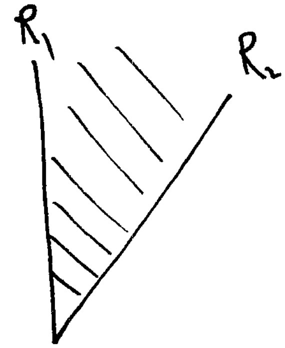
(X,Δ) klt. Δ big A ample.

MMP with scaling terminates.

Bizarre consequence:

Lemma Let $\pi: X \longrightarrow U$ be a projective morphism. $\rho(X/U) = 2$.
$K_X+\Delta$ klt. Then any sequence of flips terminates.

Pf ~~3 cases~~ $\overline{NE}(X/U) =$



3 cases

① $(K_X+\Delta)\cdot R_i \geq 0, \quad i=1,2.$
lt model. STOP.

② $-(K_X+\Delta)\cdot R_i > 0 \quad i=1,2.$
X is a Mori dream space.
There are only finitely many models. DONE

③ $(K_X+\Delta)\cdot R_i \geq 0, + (K_X+\Delta)\cdot R_{2-i} < 0.$

If we <u>ever</u> get to case ① or ② done.
ow there are never any choices.

"**Thm**" ~~Δ~~ Let (X,Δ) be a klt pair $\pi : X \to U$. ~~If $\nu(K_X+\Delta)$~~ Suppose $K_X+\Delta$ is p.e.

$K_X+\Delta+A$ is ample, A ample

Then the $(K_X+\Delta)$-MMP with scaling of A terminates iff $K_X+\Delta$ has a lt model.

Termination of Flips

James McKernan
Feb 27, 2009
11AM – 12PM

1. Intro
2. local approach
3. local global
4. global

<u>Defn</u> Let X be a variety. We say an $\mathbb{R}$-Cartier D is <u>nef</u> if $D \cdot C \geq 0$ $\forall C$ curves.

Let $\pi : X \to U$ be a projective morphism.

$$\overline{NE}(X/U) = \{ \alpha \in H_1(X, \mathbb{R}) : \alpha = \sum \lambda_i [C_i], \ \lambda_i \geq 0 \}$$
$$\pi_* \alpha = 0.$$

(X, Δ) log pairs, X normal, $\Delta = \sum b_i \Delta_i$, $b_i \in \mathbb{R}$, $b_i \geq 0$, $K_X + \Delta$ $\mathbb{R}$-Cartier
(why $\mathbb{R}$-Cartier? Want to work with $\mathbb{R}$ coeff, and want to consider the notion of homology).

$f : Y \to X$ be a <u>log resolution</u>.

$$K_Y + \tilde{\Delta} + \sum E_i = f^*(K_X + \Delta) + \sum a_i E_i$$

$\tilde{\Delta}$: strict transform of Δ

$\sum E_i$: exceptional divisor with coeff 1.

$a_i = a(X, \Delta, E_i)$ = log discrepancy of (X, Δ) w.r.t. E_i.

$a = \inf_i a_i$ = log discrepancy

(X, Δ) is log canonical if $a \geq 0$
Kawamata log terminal if $a > 0$.

$$K_Y + \Gamma = f^*(K_X + \Delta)$$
$$\Gamma = \tilde{\Delta} + \Sigma E_i - \Sigma a_i E_i$$

coeff. of Γ log canonical coeff $\Gamma \leq 1$
klt   coeff $\Gamma < 1$

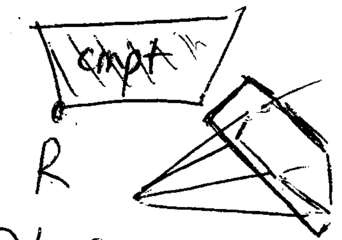
Non klt (X, Δ) = image of divisors of log discrepancy ≤ 0 (i.e., coeff ≥ 1).

<u>Intuition</u> log discrepancy is a measure of the singularities.
Smaller log disc. $\longleftrightarrow$ worse singularities

<u>MMP</u>
① Start with a klt pair (e.g. X smooth proj, $\Delta = 0$).
$K_X + D$, $\pi : X \to U$ (can take $U = pt$).

(2) Is $K_X+\Delta$ nef? If yes, then <u>stop</u>.
log terminal model (minimal model).

(3) If $K_X+\Delta$ is not nef, then $\exists$
extremal ray of $\overline{NE}(X/U)$
and a contraction



$(K_X+\Delta)\cdot R<0$,
$\phi: X\to Z$ s.t.
C is contracted iff $R=\mathbb{R}^+ \cdot C$.

(a.) $\dim Z<\dim X$ Fibers are Fano var.
Stop b/c Never nef.

(b.) ϕ is birational.

(i.) Contract a divisor. Replace X by Z, and return to (2).

(ii) ϕ is small (birat'l but doesn't contract a divisor).

There is an isom. in Codim 1.

$K_{X^+}+\Delta^+$ is ample $/Z$

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

X^+ is the flip of f.

Picard number $\rho(X^+/Z)=1$

Replace X by X^+.

E.g. $\dim X=2$

- $S=X$ sm proj. surface.

(a.) $Z=pt$, $S=X=\mathbb{P}^2$,
Z=curve, S is a $\mathbb{P}^1$-bundle over Z.

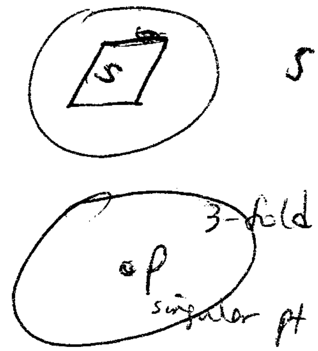
(b.) ($\nexists$ flips for surfaces so)
(i.) contract a (-1)-curve.

4/8

E.g. Take U = cone over S, general type.

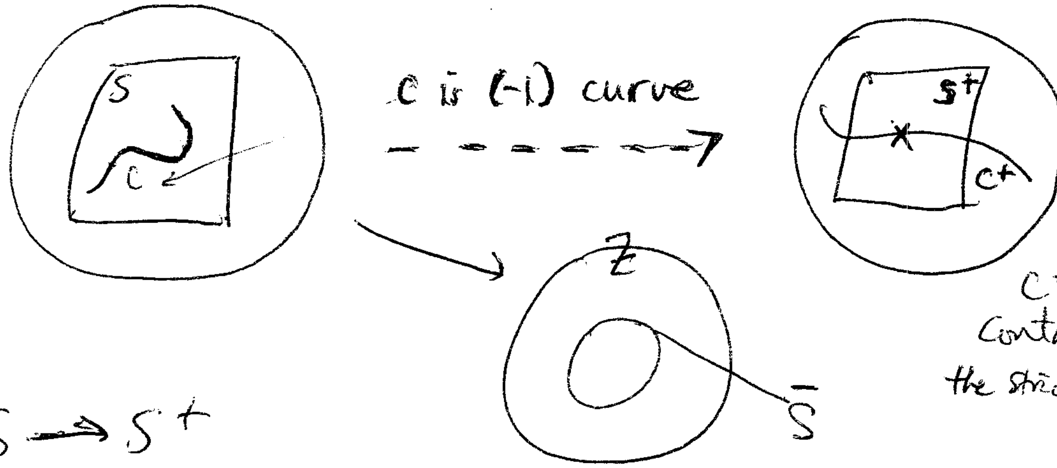
X = blow up of the cone

$$X \xrightarrow[\text{over } S.]{\mathbb{P}^1\text{-bundle}} U$$



(K_X+S)-MMP $/U$. adjunction formula $(K_X+S)|_S = K_S$

This MMP will flip all the (-1)-curves in S.



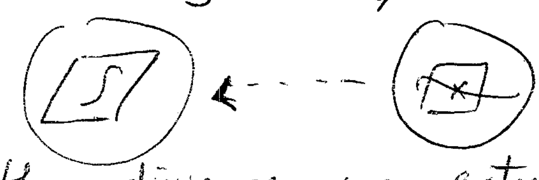
C^+ is not contained in the strict transform

$S \longrightarrow S^+$ morph. which contracts the (-1)-curves.

What is known about MMP?
Flips exist, but it is not clear
there could be an infinite sequence of
flips.

<u>Thm.</u> (special Termination)
Assume flips terminate in dim $n-1$.
The sequence of flips for $(K_X+\Delta)$-MMP
$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots$
then eventually the sequence does not
meet non klt (X, Δ).

<u>Sketch of proof</u>
Basic idea: reduce to a model where non klt
is a divisors. Assume for simplicity
this divisor is irreducible.
$(K_X+\Delta)$-MMP on $S=\lfloor \Delta \rfloor$
$(K_S+\Theta)$-MMP for some Θ.
Define Θ by adjunction:
$(K_X+\Delta)|_S = K_S+\Theta$.
One problem, $S \dashrightarrow S^+$ might
extract a divisor.



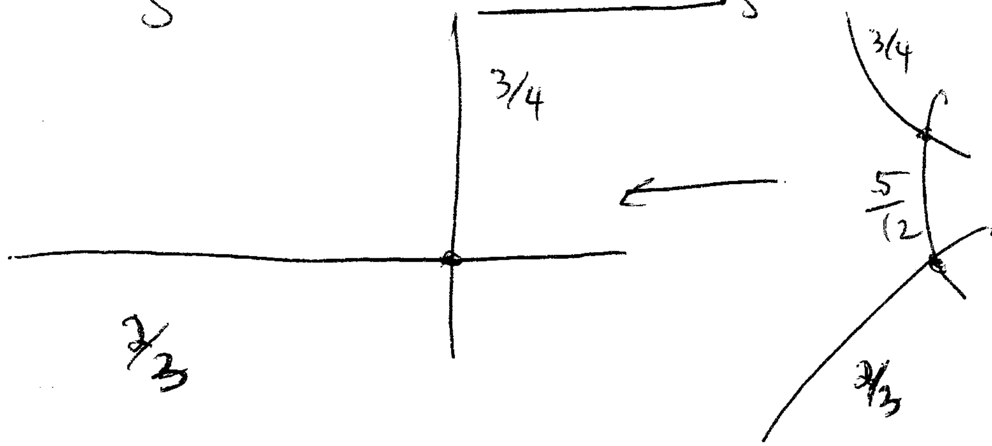
<u>Key observation</u>: the divisors we extract
must have log disc. <1 (coeff >0).
And $\exists$ only finitely many such.

<u>Extend</u> defn of <u>log disc.</u> (X, Δ), V subvariety.
$a(X, \Delta, V) = \inf\limits_i a_i(X, \Delta, E_i)$. $f(E_i) = V$.

$K_T = f^* K_S + E$

$K_T - E = f^* K_S$ $\quad S \leftarrow T$

6/8 (log disc $= n$ for n-folds)

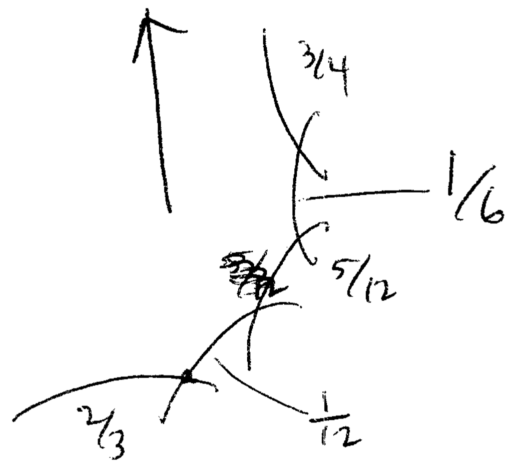


log disc $=2$ coeff

$\frac{2}{3}+\frac{3}{4}-1 = \frac{5}{12}$

$\frac{5}{12} < \frac{2}{3}, \frac{3}{4}$

S smooth surface



Conjecture (ACC)

Fix dim n. Fix I coeff set, satisfies DCC.

$\{ a(X, \Delta, x) :$ coeffs of Δ belong to I, $\dim X = n\}$

$x \in X$ satisfies ACC.

Conjecture (LSC) lower semi continuity

(X, Δ) log pair, $\forall x \in X$, $\exists$ nhbd U_x

s.t. $a(X, \Delta, x) = \inf_{y \in U} a(X, \Delta, y)$

"Log disc. is a measure of sing."

known in dim $=3$.

7/8

<u>Thm</u> (Shokurov) Assume ACC and LSC.
Then every sequence of klt flips terminates.

<u>Lemma</u> Assume LSC.
Suppose we have (X, Δ), $V \subseteq W$. Then
$$a \leq a(X, \Delta, V) \leq a(X, \Delta, W) + \operatorname{codim}(V, W)$$

<u>Observation</u> (X, Δ) W general $V \subseteq W$,
then $$a(X, \Delta, V) = a(X, \Delta, W) + \operatorname{cod}(V, W)$$

<u>Proof</u> Suppose not. Suppose we have an infinite sequence of flips
$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots$$
$$\Delta_1 \quad \Delta_2$$
Log discrepancies increase on the flips.

Assume ACC, then log discrepancy stabilizes a.
May assume dim d of V s.t. $a(X, \Delta, V) = a$ stabilizes. Consider the locus
$$W_i = \bigcup_{\substack{V_i \subseteq X_i \\ \dim(V_i) = d \\ a(X_i, \Delta_i, V_i) = a}} V_i$$
$W_i \dashrightarrow W_{i+1}$ is not an isomorphism

8/8 ✓

W_i is <u>closed</u>, and all subvarieties of W of dim d, $a(X_i, \Delta_i, V) = a$. So the map $W_{i+1} \dashrightarrow W_i$ does not extract any dim d or greater subvarieties. By intersection theory, no new cycles can be introduced.