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Kobayashi geodesic curves in $\mathcal{A}_g$.

Martin Möller and Eckart Viehweg

(Notes for a talk at the MSRI, Feb. 27, 2009)

Let $\mathcal{M}$ be a moduli stack, coarsely represented by a quasi-projective scheme M, for example the one of polarized minimal models. Consider a curve $\varphi_0 : Y_0 \to M$, factoring through the moduli stack, i.e. induced by a family $f_0 : X_0 \to Y_0$ (which we assume to have semistable reduction at $S = Y \setminus Y_0$, where Y is a smooth compactification).

Question. When is $\varphi_0 : Y_0 \to M$ "minimal"?

Tries:

A. (Differential Geometry) $\varphi_0 : Y_0 \to M$ is a geodesic (for the Hodge metric).

B. Numerical Invariants are minimal. For example Assume we have local Torelli Thm

(Analytic Geometry:) For all non-unitary sub-VHS $\mathbb{V} \subset \mathbb{W} = R^k f_{0*}\mathbb{C}_{X_0}$ there are Arakelov inequalities, and "minimal" $\leftrightsquigarrow$ those are equalities.
(Or in Topology:) (Y_0 compact) The Milnor-Wood inequalities for the Toledo invariant of all non-unitary sub VHS $\mathbb{V} \subset \mathbb{W} = R^k f_{0*}\mathbb{C}_{X_0}$ are equalities.

C. (Arithmetic geometry) Up to deformations of $\varphi_0 : Y_0 \to \mathcal{M}$, the curve Y_0 is a Shimura curve (i.e. given by linear algebra). Up to deformations

Example. I. (Very classical moduli.) $\mathcal{M} = \mathcal{A}_g$ = the moduli stack of polarized abelian varieties.

Known. (Abdulali, Moonen, Viehweg-Zuo) For $\mathcal{M} = \mathcal{A}_g$ (and $k = 1$) the conditions A), B) and C) are equivalent.

II. $\mathcal{M} = \mathcal{M}_g$ = moduli stack of curves.

Known. (Möller-Viehweg-Zuo) There are no such curves, except for $g = 1$ and 3.

III. $\mathcal{M} = \mathcal{M}_g^{ct}$ = moduli stack of stable curves of compact type.

Coleman Conjecture. $\rightsquigarrow$ Such curves should only exist for finitely many values of g ($g \leq 7$???).

IV. $\mathcal{M}$ = moduli of polarized CY-n-folds. Very little known.

Known. (Viehweg-Zuo) Few results: No compact curves $Y_0 = Y$ for n odd.

(Viehweg-Zuo, Borcea, Voisin, Rohde) Non-compact examples for all n.

Today I will concentrate on the case $\mathcal{M} = \mathcal{A}_g$, but I will replace "Hodge metric" by "Kobayashi metric". Such curves even exist in $\mathcal{M}_g$, for all g.

We will allow ourselves to replace Y_0 by étale coverings, so we may as well assume that $\mathcal{A}_g$ is a fine moduli scheme (with some level structure).

Definition: $\mathcal{D}$ complex domain.

I. Kobayashi pseudo-distance

$$d(p,q) = \mathrm{Inf}\Big\{ \sum_{i=1}^{\ell} d_i(a_{i-1}, a_i);\ a_{i-1}, a_i \in \Delta_i,\ a_0 = p,\ a_\ell = q \Big\}$$

where Δ_i is a disc and d_i the Poincaré metric.

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II. $\Delta \subset \mathcal{D}$ is a *(complex) Kobayashi geodesic* if the restriction of the Kobayashi metric on $\mathcal{D}$ to Δ coincides with the Poincaré metric.

III. We call a map $\varphi_0 : Y_0 \to A_g$ *a Kobayashi geodesic*, if its universal covering map $\widetilde{\varphi}_0 : \widetilde{Y}_0 \cong \Delta \to \widetilde{A}_g = \mathbb{H}_g$ is a Kobayashi geodesic.
upper half plane

In spite of the "topological" definition, the "Kobayashi geodesics in $\mathcal{A}_g$" seem to be an algebraic notation.

Theorem: (Möller-V.) The following conditions are equivalent:

1) Y_0 is a Kobayashi geodesic.

2) The variation of Hodge structures $\mathbb{W} = \mathbb{W}_{\mathbb{Q}} \otimes \mathbb{C}$ contains some non-unitary subvariation of Hodge structures $\mathbb{V} \neq 0$, which satisfies the Arakelov equality.

3) $\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}}) \to \Omega^1_Y(\log S)$ splits.

Here:

- $\overline{\mathcal{A}}_g$ denotes one of the toroidal compactifications with boundary $S_{\overline{\mathcal{A}}}$ (Mumford et al.), and $\varphi : Y \to \overline{\mathcal{A}}_g$ an extension of φ_0 to Y.
$S_{\bar{A}_g} = \bar{A}_g - A_g$
- (Arakelov inequality) $\mathbb{V} \subset \mathbb{W} = R^1 f_{0*}\mathbb{C}_{X_0}$ is a sub local system, non-unitary $(\theta \neq 0)$

$\rightsquigarrow$ Deligne extension $\mathcal{H}$ of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_{Y_0}$ with $F^\bullet$ filtration.

$\rightsquigarrow$ logarithmic Higgs bundle $E = \mathfrak{gr}_{F^\bullet}(\mathcal{H}) = E^{1,0} \oplus E^{0,1}$ with Higgs field
$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S).$$

Arakelov inequality (due to Faltings/Deligne)
$$(*) \qquad \frac{\deg(E^{1,0})}{\mathrm{rk}(E^{1,0})} - \frac{\deg(E^{0,1})}{\mathrm{rk}(E^{0,1})} \leq \deg(\Omega^1_Y(\log S)).$$

Lemma: (V.-Zuo) The following are equivalent:

a) Arakelov equality (i.e. $(*)$ is an equality).

b) "maximal Higgs" (i.e. θ is an isomorphism).

c) $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$ with $\mathbb{U}$ unitary and $\mathbb{L}$ of rank two, with Higgs field
$$\mathcal{L} \xrightarrow{\cong} \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S),$$
hence $\mathcal{L}$ is a logarithmic theta-characteristic.

Examples:

a) All Shimura curves. (Geodesics for the Hodge metric $\Rightarrow$ Kobayashi geodesic)

b) $Y_0 \subset \mathcal{M}_g \Longrightarrow Y_0$ is affine and a Teichmüller curve (Möller, McMullen).

c) **Open Question:** Are there compact Kobayashi geodesics in $\mathcal{M}_g^{ct}$ for infinitely many g?

d) There are Kobayashi geodesics $Y_0 \in \mathcal{A}_g$ that are neither Shimura nor Teichmüller curves. (Use classification of triangle groups $\Gamma = \Delta(l, m, n)$).

e) Typical example: Curves on g-dimensional Hilbert modular varieties Z_0 $\subset A_g$ (g-fold) such that on the universal coverings $\Delta = \widetilde{Y}_0 \to \widetilde{Z}_0 = \Delta^g$ at least one projection
$$\begin{tikzcd} \Delta = \widetilde{Y}_0 \arrow[r] \arrow[rr, bend right, "\cong"'] & \widetilde{Z}_0 = \Delta^g \arrow[r, "pr"] & \Delta \end{tikzcd}$$
induces an isometry.

Corollaries of the Theorem: Assume Y_0 is an **affine** Kobayashi geodesic and that $\mathbb{W}_{\mathbb{Q}}$ **is irreducible** ($\leftrightsquigarrow$ general fibre of f_0 is simple). Then (up to étale coverings of Y_0)

$= R^1 f_* \mathbb{Q}_{X_0}$

exercise.

I) For some totally real number field F $\mathbb{W}_F = \mathbb{W}_{\mathbb{Q}} \otimes F = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g$, with $\mathrm{rk}(\mathbb{L}_i) = 2$. Each $\mathbb{L}_i$ is a VHS of weight one, non unitary, and $\mathbb{L} = \mathbb{L}_1$ is maximal Higgs. ($\Longrightarrow \mathbb{L}_i$ is conjugate to $\mathbb{L}_1$ and $\mathbb{W}_{\mathbb{Q}}$ is the Weil restriction of any $\mathbb{L}_i$).

* II.) Y_0 lies on a g-dimensional Hilbert modular variety g-fold (as in Example e)).

III.) $Y_0 \subset \mathcal{A}_g$ is defined over a number field $\overline{\mathbb{Q}}$.

On the proof of the Theorem. $g = [F : \mathbb{Q}]$

As in III) Kobayashi geodesics are related to polydiscs:
Via the Cayley transformation $\mathbb{H}_g$ has a realization as a bounded symmetric domain

$$\mathcal{D}_g \subseteq \{Z \in M^{g\times g}(\mathbb{C}) \,|\, I_g - ZZ^* > 0\} = \{Z \in M^{g\times g}(\mathbb{C}) \,|\, ||Z|| < 1\} \subset \underbrace{\Delta^{g^2}}_{\text{bd domain}} \subset \mathbb{C}^{g^2},$$

where $||Z||$ is the operator norm. A typical polydisc Δ^g is given by the intersection of $\mathcal{D}_g$ with the diagonal in $\mathbb{C}^{g^2}$.

Example (*): Choose for some $\ell \leq g$ a multidisc Δ^ℓ, a totally geodesic embedding $i : \Delta^\ell \times \mathcal{D}_{g-\ell} \to \mathcal{D}_g$ (for the Bergmann metric), and $\widetilde{\psi}_0 : \Delta \to \mathcal{D}_{g-\ell}$ (not a geodesic). Then $i \circ (\mathrm{diag}, \widetilde{\psi}_0)$ defines a Kobayashi geodesic.

Down to earth: Consider block-matrices

$$\tau \mapsto \begin{pmatrix} \tau \cdot E_\ell & 0 \\ 0 & Z_{g-\ell}(\tau) \end{pmatrix},$$

with $||Z_{g-\ell}(\tau)|| < |\tau|$, where E_ℓ = $\ell \times \ell$ identity matrix.

Fact: (M. Abate) Up to conjugation with $U \in U(g)$ all Kobayashi geodesics are of this form.

Starting point. Let K be the fix group of a point 0 for the $\mathrm{Sp}(2g, \mathbb{R})$ action. Then K acts on the set of polydiscs, and

$$\mathcal{D}_g = \bigcup_{k\in K} k(\Delta^g).$$

Corollary: If $\Delta = \widetilde{Y}_0$ is the universal covering of $Y_0 \to \mathcal{A}_g$, hence

$$\pi_1(Y_0, *) \subset H = \{\sigma \in \mathrm{Sp}(2g, \mathbb{R});\ \sigma(\widetilde{Y}_0) = \widetilde{Y}_0\}.$$

Then the multidisc $i_0 : \Delta^\ell \to \mathcal{D}_g$ is H invariant, and the block decomposition of $\widetilde{\varphi}_0$ is preserved by the action of H.

Corollary: There is a splitting of $\varphi^*\Omega^1_{\mathcal{A}_g} \to \Omega^1_{Y_0}$.

To extend it to the boundary, one uses Mumford's explicit description of a toroidal compactification $\overline{\mathcal{A}}_g$. So one gets

1) Y_0 Kobayashi geodesic $\overset{*}{\Longrightarrow}$ 3) $\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}}) \to \Omega^1_Y(\log S)$ splits.

To see that the splitting implies the existence of some $\mathbb{V} \subset \mathbb{W}$ which is maximal Higgs, as claimed in 2), one uses the description of $\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}})$ in terms of Higgs bundles:

If $\mathbb{W} = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_\ell$ is the decomposition in $\mathbb{C}$-irreducible direct factors $\mathbb{V}_i$, with Higgs bundles $(E_i^{1,0} \oplus E_i^{0,1}, \theta_i)$, then $(E_1^{1,0} \otimes E_1^{0,1}) \oplus \cdots \oplus (E_\ell^{1,0} \otimes E_\ell^{0,1})$

$$\varphi^*\Omega^1_{\overline{\mathcal{A}}_g}(\log S_{\overline{\mathcal{A}}}) = S^2(E_1^{1,0} \oplus \cdots \oplus E_\ell^{1,0}) \subseteq (E_1^{1,0} \oplus \cdots \oplus E_\ell^{1,0}) \otimes (E_1^{0,1} \oplus \cdots \oplus E_\ell^{0,1})$$

So there is some $\mathbb{V} = \mathbb{V}_i$, with Higgs bundle $(E^{1,0} \oplus E^{0,1}, \theta)$, responsible for the splitting.

$E_i^{1,0} \otimes E_j^{0,1}$ goes to 0 in $\Omega^1_Y(\log S)$

$\Omega^1_Y(\log S) \rightleftarrows E_1^{1,0} \otimes E_1^{0,1}$

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Study the behavior of the Harder-Narasimhan filtrations of $E^{1,0}$ and $E^{0,1}$ under θ.

Some elementary calculations and Simpson's correspondence imply that the Harder-Narasimhan filtrations can only have one step, hence that $E^{1,0}$ and $E^{0,1}$ are both semistable.

Then $E^{1,0} \otimes E^{0,1\vee}$ is semistable, hence its slope coincides with the one of the direct factor $\Omega^1_Y(\log S)$. This is the Arakelov equality, and as we stated already, it implies "maximal Higgs".

Finally "2) $\Longrightarrow$ 1)" is an easy exercise in calculating period matrices.

For Corollary I. and II. (Here Y_0 is affine, hence $S \neq \emptyset$)

The existence of a point with non-trivial unipotent monodromy for $\mathbb{V}$ implies that

i) $\mathbb{V}$ can not deform ($\Rightarrow$ defined over $\bar{\mathbb{Q}}$)

ii) And no conjugate of $\mathbb{V}$ can be unitary.

i)&ii) imply that the maximal unitary sub local system of $\mathbb{W}$ is defined over $\mathbb{Q}$, hence trivial (after étale covering of the base).

So the irreducibility of $\mathbb{W}_{\mathbb{Q}}$ implies that there is no such sub local system.

Similar arguments applied to the endomorphisms of $\mathbb{V} = \mathbb{L} \otimes \mathbb{U}$ shows that $\mathbb{U} = \mathbb{C}^r$, and finally a little bit of classification of "generalized Hilbert modular surfaces" shows that $\mathbb{V} = \mathbb{L}$ and that this is defined over a totally real number field F. Taking the Weil-restriction one obtains $\mathbb{W} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g$ with the properties stated.

For Corollary III.

Firstly: (Does not need "Y_0 is affine") The characterization of Kobayashi geodesics by 3) is "algebraic". Hence if it holds over some function field, it extends to an open parts of the corresponding family.

But if one has a smooth family over a higherdimensional base, say $h : \mathcal{Y} \to U$ with boundary $\mathcal{S}$, étale over U, and if $\mathcal{Y}_0 = \mathcal{Y} \setminus \mathcal{S} \to \mathcal{A}_g \times U$ is generically finite, then the splitting implies that

$$0 \to h^*\Omega^1_U \to \Omega^1_{\mathcal{Y}}(\log \mathcal{S}) \to \Omega^1_{\mathcal{Y}/U}(\log \mathcal{S}) \to 0$$

splits as well, hence that $h : \mathcal{Y} \to U$ is locally analytically trivial.

Secondly: All $\mathbb{L}_i$ non-unitary $\Longrightarrow$ $\mathrm{End}(\mathbb{W})^{0,0} = \mathrm{End}(\mathbb{W})$ and this (by Faltings) $\Longrightarrow$ rigidity.

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Kobayashi geodesics in A_g

([redacted] j.w.w. Martin Möller)

Eckart Viehweg
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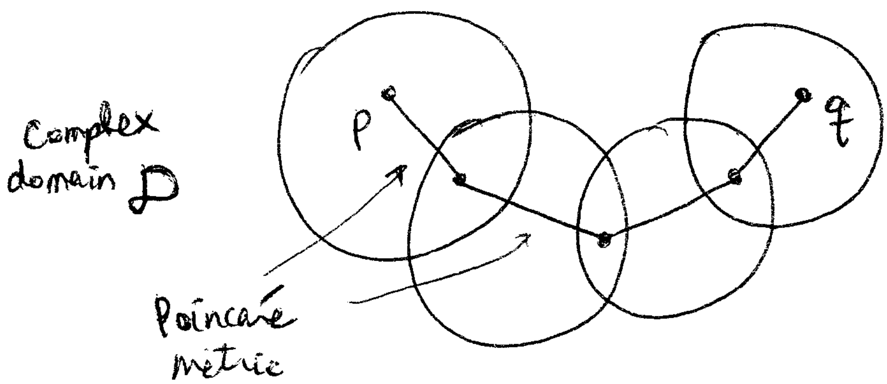
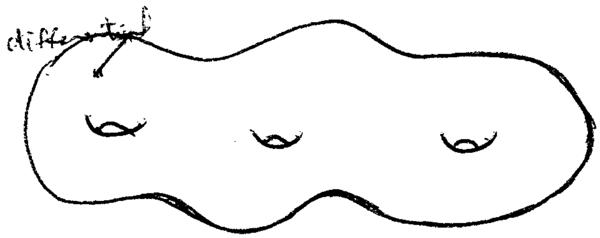
Kobayashi metric



$$d(p,q) = \inf \left\{ \sum_{i=1}^{l} d_i(a_{i-1}, a_i) : a_{i-1}, a_i \in \Delta_i, \ a_0 = p, \ a_l = q \right\}$$

Δ_i disc, $\quad d_i =$ Poincaré metric

Teichmüller curve (lies in M_g)



On the proof of the thm.

$Y \not\ni Y_0$, $s \in S$

$\mathbb{L} \otimes \mathbb{U} \quad \cancel{\oplus \, \Pi}$

unitary

$$\begin{tikzcd} Y_0 \arrow[r, "\text{generically finite}"] \arrow[d] \arrow[dr, equal] & A_g \\ \mathbb{U} & Y_0 \times \mathbb{U} \arrow[l] \end{tikzcd}$$

Faltings $\quad \dim \mathbb{U} = 0$