1 Lev Borisov, 23 March 2009

Title: Strong exceptional collections of line bundles on Fano toric DM stacks

Keywords: exceptional collection, Fano, toric Deligne-Mumford stack, derived category, toric del Pezzo DM stack, forbidden cone

Summary: Full strong exceptional collections of coherent sheaves are very useful in working with the derived category, but it is rare that a variety or stack has a full strong exceptional collection. This talk discusses a conjecture that every smooth toric nef-Fano DM stack possesses such a collection, and proves it for such stacks of dimension at most two or Picard number at most two.

Joint work with Zheng Hua, to appear in Advances in Math.

Motivation Let X be a smooth projective complex variety. Denote by $D(X)$ the bounded derived category of coherent sheaves: complexes of sheaves on X up to quasi-isomorphism. (This contains a lot of information on K-theory, etc.)

In "nice cases" $D(X)$ has a full strong exceptional collection of line bundles. This is in fact the best possible case. For quadrics, can have full strong exceptional collection but not of line bundles. For most X, nothing nearly as nice.

Example: $X = \mathbb{P}^n$. Then line bundles $\mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O}$ are a full strong exceptional collection:

- $Ext^{>0}(L_i, L_j) = 0$ ("strong exceptional" collection)
- These generate $D(X)$ under shifts, cones, etc.: the minimal triangulated full subcategory of $D(X)$ containing these L_i is $D(X)$ itself ("full" collection)

$D(X) = D^b$(modules over $End(X, \oplus L_i)$). (Aside: we can think of this in terms of a quiver.)

Question: when can we expect to have full strong exceptional collections? King conjectured for all toric varieties; Perling came up with counter-examples in dimension 2. A refined conjecture posits that it is true for all Fano toric varieties, or nef Fano toric varieties. Costa and Miro-Roig have proved that it is true for toric varieties with $rk\ Pic \leq 2$, Fano or not.

In principle we could just check example by example, but it feels more natural to start talking about smooth Fano toric Deligne-Mumford stacks rather than varieties.

Theorem 1. *Every smooth toric del Pezzo Deligne-Mumford stack has a full strong exceptional collection of line bundles.*

Audience Q: is this known for del Pezzo surfaces in general? A: I think so. Q: do you know what they are on the cubic surface? A: There's more than one, and I don't know off the top of my head. Start from $\mathbb{P}^2$ and blow up, building up from there.

Today, we'll go through a brief description of the argument of the paper. The arguments are mainly from complex geometry; not so much emphasis on toric/stack part.

Back to $\mathbb{P}^n$, the inspiration for the argument: why do these line bundles generate the whole category?

- Derived category of $\mathbb{P}^n$ is generated by line bundles by Hilbert's Syzygy theorem.
- on $\mathbb{P}^n$ there is Koszul complex

$$0 \to \mathcal{O}(-n-1) \to \ldots \to \mathcal{O}^{\oplus\binom{n+1}{2}}(-2) \to \mathcal{O}^{\oplus(n+1)}(-1) \to \mathcal{O} \to 0$$

  . Can twist by k to move right or left, and every line bundle can be written in terms of these guys "in the window." See picture at end of pdf for illustration of "window."

- On a smooth toric stack Σ, $D(\mathbb{P}_\Sigma)$ is generated by line bundles. (Feature of toric situation: don't see this for Grassmannians, for instance.) So once again we'll be looking for a "window."

(Note prompted by audience question: Fano does not appear here yet: it is relevant but not clear how Fano relates.)

In this window, no two line bundles have Exts between each other. The first case where things get funny is when the rank of the Picard group is 3 and dimension is 2. Let N be a fan with five rays generated by vectors $v_1, \ldots, v_5$. Fano condition forces the pentagon (convex hull of v_i) to be convex. $E_1, \ldots, E_5 \in Pic(\mathbb{P}_\Sigma)$, $\sum_i (m \cdot v_i)E_i = 1 \; \forall m \in \mathbb{N}^*$. (Remember, we're trying to construct a polytope giving line bundles with no Exts between them.)

Let L be a line bundle. How do we check acyclicity? Step back:

- $H^0(L) \neq 0$ is equivalent to $\sum a_i E_i \cong L$ for $a_i \in \mathbb{Z}_{\geq 0}$
- $H^{top}(L) \neq 0$ equivalent to $L \cong \sum a_i E_i$ for $a_i \in \mathbb{Z}_{\leq -1}$ by Serre duality.
- Similarly, all the rest of the $H^k(L) \neq 0$ are governed by sign patterns on coefficients: $H^k(L) \neq 0$ is equivalent to $L \cong \mathcal{O}(\sum a_i E_i)$ such that the set I of i with $a_i \leq -1$ induces a simplicial complex with nontrivial $(k-1)$st reduced homology.

For all $I \subseteq \{1, \ldots, n\}$ consider the simplicial complex C_I built from collections of indices in I that are also encoding cones in Σ. If reduced homology of C_I is nontrivial ($\neq 0$), then $H^\bullet(\sum_{i\in I} a_i E_i + \sum_{j \notin I} a_j E_j) \neq 0$ where $a_i \geq 0$ for is and $a_j \leq -1$.

Definition 1. *For each I with nontrivial reduced homology, define the forbidden cone inside $Pic_{\mathbb{R}}$: $-\sum_{i\in I} E_i + \sum_{i \in I} \mathbb{R}_{\geq 0} E_i + \sum_{i\notin I} \mathbb{R}_{\geq 0}(-E_i)$. If your line bundle falls inside the cone, there is a chance of nonzero cohomology.*

Definition 2. *If L is outside all forbidden cones, then it is called strongly acyclic (and in particular it is acyclic).*

The geometric meaning of strongly acyclic is not quite clear yet.

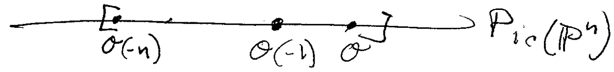
Q: i can run over 1 through 5 in our previous example? A: Yes; but if a subset in I is chosen so that the hull of the v_i are a line segment rather than a two-dimensional simplicial complex then there is no forbidden cone so it's not interesting. Are there any ways that the forbidden cone can contain everything? This is ruled out in Fano case. (*Did I hear this correctly?)

Start drawing pictures: pictures at end of this pdf. $rk\ Pic = 3$. $\hat{Pic}_{\mathbb{R}} = Pic_{\mathbb{R}}/\mathbb{R}\sum E_i$. Then have $\hat{E}_1, \ldots, \hat{E}_5$ in $\hat{Pic}_{\mathbb{R}}$.

Projections of forbidden cones to $\hat{Pic}$: some more *pictures.* Draw this 10-gonal prism in Pic, which is our window mentioned above. As we move it around, it gives sets of line-bundles. Q; is Grothendieck group also generated by window? A: Yes, that's the expectation. Q: Fano only came in in convexity, right? A: yes.

In dimension 3, can construct same Q, but then problems occur. $\hat{P}$ has to have centrally symmetric facets, with vertices of Q hitting midpoint of each facet. Are there any polytopes other than zonotopes that do this? So this is open for dimension 3, rank of Picard ≥ 3. If anyone knows about centrally symmetric facets, talk to me!

Q: What do we want from $\hat{P}$? A: Q is Minkowsi sum of E_is and $\hat{P}$ goes around it. (picture 4 on paper)



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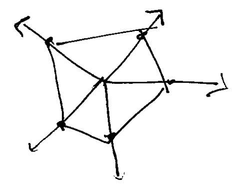
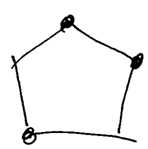
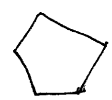




Start drawing



$\mathrm{rk}\,\mathrm{Pic} = 3$

$\widehat{\mathrm{Pic}}_{\mathbb{R}} = \mathrm{Pic}_{\mathbb{R}} / \mathbb{R}(\sum_i E_i)$

modding out is sort of like looking at local Calabi-Yau

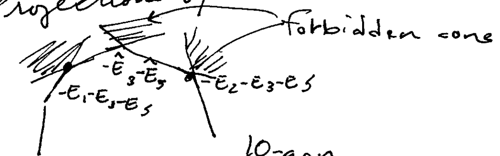
$\cancel{\hat{v}_1 \dots \hat{v}_5}$

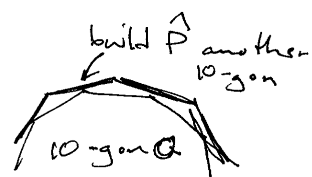
$\hat{E}_1 \dots \hat{E}_5$ s.t.

$\sum (m \cdot v_i) \hat{E}_i = 0$

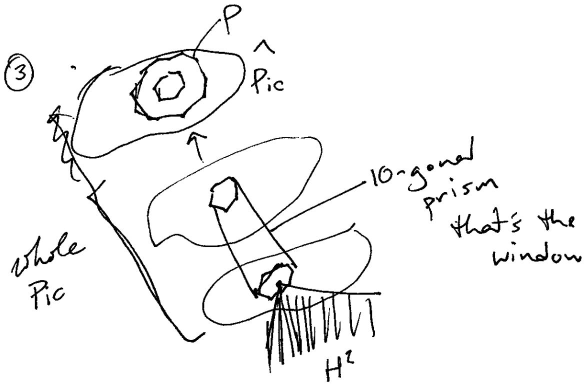
$\sum \hat{E}_i = 0$

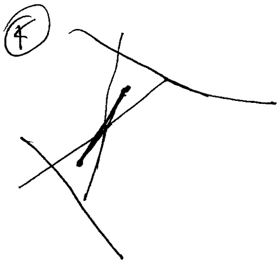
Projections of forbidden cones

① 

② 

$\sum_{i=1}^{5+5} [\mathcal{O}_i - \hat{E}_i]$

③ 

④ 

3/23 L. Borisov
(w/ Z. Hua)

Background: X = sm proj. variety / $\mathbb{C}$

$\mathcal{D}(X) = \left(\begin{array}{l} \text{category of bdd complexes of sheaves on } X \\ \text{up to quasi-isomorphism} \end{array} \right)$

In "nice cases," $\mathcal{D}(X)$ has
a full strong exceptional collection of line bundles

$= \{\mathcal{L}_1, \dots, \mathcal{L}_{n+1}\}$

Ex: $X = \mathbb{P}^n$. $\{\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}\}$ has properties:

- $\mathrm{Ext}^{>0}(\mathcal{L}_i, \mathcal{L}_j) = 0$ ("strong exc. collection")

- The $\mathcal{L}_i$ generate $\mathcal{D}(X)$

$\rightsquigarrow \mathcal{D}(X) = \mathcal{D}^b(\text{modules over } \mathrm{End}(X, \oplus \mathcal{L}_i))$

Question: When does one have a f.s.e.c.?

King: all toric varieties?

Perling: gave counterexample

Refinement of conjecture: all Fano toric varieties?

Known: for X = toric, $\mathrm{rk}\,\mathrm{Pic}(X) \leq 2$. [

<u>Thm</u> (B.-Hua): Every smooth toric del Pezzo DM stack has a f.s.e.c. of line bundles.

(Goal: describe argument, give open questions)

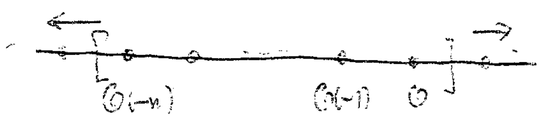
<u>Ex</u>: $X = \mathbb{P}^n$. Hilbert syzygy thm $\Longrightarrow D(\mathbb{P}^n)$ gen'd by line bundles.

Have Koszul cx:

$$0 \longrightarrow \mathcal{O}(-n-1) \to \cdots \longrightarrow \mathcal{O}(-2)^{\oplus \binom{n+1}{2}} \longrightarrow \mathcal{O}(-1)^{\oplus (n+1)} \longrightarrow \mathcal{O} \to 0.$$

Can twist by $\mathcal{O}(k)$, so add k throughout.

$\Longrightarrow$ Knowing $n+1$ consecutive LB's, get all others.

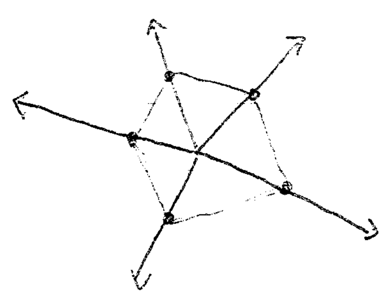


"window" in $\mathrm{Pic}(\mathbb{P}^n)$.

<u>Fact</u>: On a smooth toric stack $\mathbb{P}_\Sigma$, $D(\mathbb{P}_\Sigma)$ is gen'd by LB's.

<u>Case</u> $\dim X = 2$, $\mathrm{rk}\,\mathrm{Pic}\, X = 3$.

Lattice N, fan:



Generators $v_1, \ldots, v_5$ (not nec. primitive), s.t. pentagon is convex. (Fano condition)

Pic is gen'd by $E_1, \ldots, E_5 \in \mathrm{Pic}(\mathbb{P}_\Sigma)$, mod $\sum_i \langle m, v_i \rangle E_i = 0$ for $m \in N^*$.

Let L be a line bundle. How to check acyclicity?

- $H^0(L) \neq 0 \iff L \sim \sum a_i E_i$, $a_i \in \mathbb{Z}_{\geq 0}$.
- $H^{top}(L) \neq 0 \iff L \sim \sum a_i E_i$, $a_i \in \mathbb{Z}_{\leq -1}$ (Serre duality)

For any $I \subseteq \{1, \dots, n\}$ ($n=5$ in this case), consider simplicial cx built from collections of indices in I that are cones in Σ; call this cx C_I.

~~Thm~~ If reduced homology of C_I is $\neq 0$, then $H^0\left(\sum_{i \in I} a_i E_i + \sum_{j \notin I} a_j E_j\right) \neq 0$

$(a_i \geq 0)$ $(a_j \leq -1)$

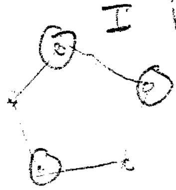
(By standard arguments.)

For each I s.t. C_I has nontrivial reduced homology, define the <u>forbidden cone</u> in $\mathrm{Pic}_{\mathbb{R}}$ to be

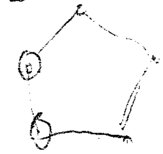
$$\mathbb{R}_{\geq 0} \cdot \left(-\sum_{j \notin I} E_j\right) + \sum_{i \in I} \mathbb{R}_{\geq 0} E_i + \sum_{j \notin I} \mathbb{R}_{\geq 0}(-E_j) .$$

<u>Def'n</u> If L is outside all forbidden cones, then it is called <u>strongly acyclic</u>. (Such L are necessarily acyclic.)

Ex: I has nontriv. homology

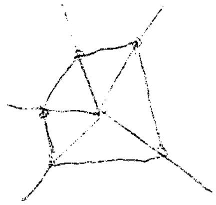


I has trivial reduced homology



$\Rightarrow$ no forbidden cone.

Δ :



rk Pic $= 3$

Let $\widehat{\mathrm{Pic}}_{\mathbb{R}} = \mathrm{Pic}_{\mathbb{R}} / \mathbb{R}\cdot(\sum E_i)$;

write $\hat{E}_1, \dots, \hat{E}_5$ for images.

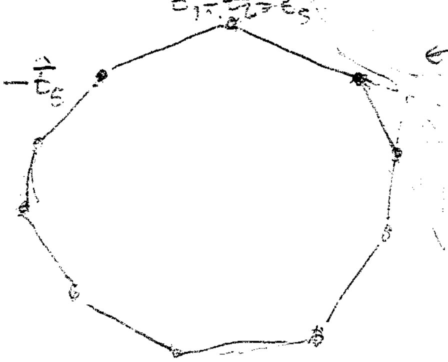
$\rightsquigarrow \quad \sum \langle m, v_i \rangle \hat{E}_i = 0 , \quad \sum \hat{E}_i = 0 .$

Projections of forbidden cones to $\widehat{\mathrm{Pic}}$:

natural 10-gon :

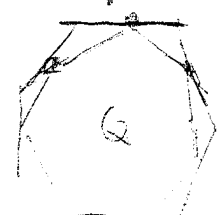
$-\hat{E}_1 - \hat{E}_2 - \hat{E}_5$

$-\hat{E}_1 - \hat{E}_3 - \hat{E}_5$



forbidden cones

$\hat{P}$



10-gon $Q \cong \sum [0, -\hat{E}_i]$

$\hat{P}$ = "window" [expect it to generate Groth. gp?]

Over $\hat{P}$, in Pic, get 10-gonal prism. Lattice pts give f.s.e.c.

Q: Are there polytopes which aren't zonotopes with centrally symmetric facets?

(Zonotope = Minkowski sum of segments)