1 Nicolas Perrin

Title: Towards a Littlewood-Richardson rule in the Kac-Moody setting

Keywords: cominiscule homogeneous space, Littlewood-Richardson, jeu de taquin, heap

Summary: Everyone loves the Littlewood-Richardson rule. Thomas and Yong generalised it to cominiscule homogeneous spaces, and the speaker and P.E. Chaput generalised it further to compute c_{uv}^w for w a Λ-cominiscule element. The introduction includes explanation of Jeu de Taquin and miniscule/cominiscule.

Joint work with P.E. Chaput.

Fix notation: G will be a Kac-Moody group. (Think of a semisimple group or even SL_n if you don't know what that is.) T maximal torus, B Borel subgroup, P maximal parabolic subgroup, $T \subset B \subset P \subset G$. (While we assume P maximal for this talk, the results presented here hold for any P.)

W Weyl group of G, W_P Weyl group of P. $X = G/P$ an ind-variety, and we are interested in computing $H^*(G/P, \mathbb{Z})$. Recall that

$$G/P = \coprod_{w \in W/W_P} BwP/P.$$

This is the Bruhat decomposition; it's a cellular decomposition and yields a basis for homology: Define closures $\overline{BwP/P} = X(w)$ (Schubert varieties) and $[X(w)] = \sigma_w \in H_*(G/P, \mathbb{Z})$. Then $(\sigma_w)_{w \in W/W_P}$ form a basis of H_*, and (σ^w) a dual basis in H^*. Finally, $\sigma^u \cup \sigma^v = \sum c_{uv}^w \sigma^w$ give Littlewood-Richardson coefficients.

Problem: Give a combinatorial formula for c_{uv}^w.

The most famous answer to this problem is the LR-rule using Jeu de Taquin. Thomas and Yong generalized it to all miniscule/cominiscule varieties, and our results generalize these rules.

1.1 Jeu de Taquin

P a poset which bounded below: $\forall x \in P$, $\#\{y \in P | y \leq x\} < \infty$.

Definition 1. *An order ideal in P is a finite $I \subset P$ st for $x \in I$, $(y \leq x \Rightarrow y \in I)$.*

We let $I(P)$ denote the set of order ideals in P.

A standard tableau T in P is a pair (ν, λ) of order ideals $\lambda \subset \nu$ and an order preserving bijection $\nu - \lambda \to [1, \#(\nu - \lambda)]$. (ν, λ) is shape of T.

Example: look at *picture* at end of file.

We let $jdT(T) = T'$. $sh(T') = (*, \phi)$.

Definition 2. *A jdt slide is the following procedure: for T standard tableau of shape (ν, λ), choose a vertex $x \in \lambda$. Take the element y in ν that covers x with the minimal label.*

Do jdt slides until there is no vertex with empty label covered by a labeled vertex. Example: pictures. Choice made in first part is important: jdt is not well-defined (see pic).

Definition 3. *A poset has jdt property if jdt is well-defined.*

(What does this have to do with computing cohomology? There's a combinatorial framework where posets with jdt property appear as parametrising families of cohomology classes.)

1.2 Λ-cominiscule elements

There are natural representations of W/W_P given by $W^P \subset W$, where W^P is the set of minimal length representations of elements in W/W_P.

For P maximal parabolic subgroup, let Λ be its associated weight. (Take minimal embedding of $G/P \subset \mathbb{P}\,V(\Lambda)$ where $V(\Lambda)$ is the representation of G of highest weight Λ.)

Definition 4. *An element $w \in W$ is Λ-miniscule if $\exists$ a reduced expression $w = s_{i_1} \dots s_{i_\ell}$, where s_{i_k} is the reflection across the simple root α_{i_k}, such that for all k, $s_{i_k} \cdot s_{i_\ell}(\Lambda) = s_{i_{k+1}} \cdots s_{i_\ell}(\Lambda) - \alpha_{i_k}$.*

$| < s_{i_{k+1}} \cdots s_{i_\ell}, \alpha_{i_k} > | = 1$. Cominiscule = miniscule for $\Lambda^\vee$. We will skip this for now.

Fact: A Λ-miniscule element is $\subset W^P$.

Definition 5. *(Also Proposition) G/P is miniscule if and only if any element in W^P is Λ-miniscule.*

Definition 6. *Take $\tilde{w}$ a reduced expression of w. The heap of $\tilde{w}$, $H(\tilde{w})$, is the set $[1, \ell]$ with the order p smaller than q if $p > q$ and s_{i_p} and s_{i_q} do not commute.*

Theorem 1. *(Stembridge, Proctor) If w is Λ-miniscule then*

1. *$H(\tilde{w}$ does not depend on the reduced expression of w.*
2. *$[e, w] \mapsto$(order ideals in $H(w)$) is order-preserving bijection, $u \mapsto H(u)$.*
3. *$H(w)$ has the jdt property.*

In particular as a corollary of this result we get

Corollary 1. *For $u, v \leq w$ Λ-miniscule elements and U a fixed tableau of shape $(H(v), \emptyset)$, the number of standard tableaux T with shape $(H(w), H(u))$ such that jdt(T)= U does not depend on U. So we many define t^w_{uv} to be this number.*

Conjecture 1. *If u, v, w are Λ-miniscule, then $c^w_{uv} = t^w_{uv}$.*

Evidence:

Theorem 2 (Thomas-Yong)**.** *The conjecture is true for X miniscule.*

The main theorem, then, of Chaput and Perrin is

Theorem 3. *The conjecture is true for G/P with G finite dimensional algebraic group.*

Before discussing idea of proof, some applications:

Definition 7. *G/P is an adjoint variety if it is isomorphic to the closed orbit of G in $\mathbb{P}\,\mathrm{Lie}(G)$.*

Proposition 1. *When G/P is adjoint,* $\dim(G/P) = 2r + 1$ *(note: odd!). If degree $\sigma^w \leq r$, then w is Λ-cominiscule.*

For classical groups, work of Buch, Kresch, Tamvakis has already given this, but this is new for exceptional groups. There is software for computing products in the quantum cohomology of these varieties in our webpages.

In quantum cohomology, we get

Theorem 4 (Chaput-P)**.** *For X adjoint, $QH^*(G/P)$ is semi-simple if and only if G is not simply laced.*

(In that case we get an involution sending Gromov-Witten invariants of high degree to Gromov-Witten invariants of low degree.)

1.3 Sketch of Proof

Strategy is similar to that of Thomas and Yong.

We prove that c_{uv}^w and combinatorial invariants t_{uv}^w satisfy similar properties.

- compatible with Chevalley formula
- recursion formulas
- t_{uv}^w define an algebra structure on $H^*_{min}(G/P, \mathbb{Z})$

First two ideas are found in work of Yong-Thomas, last added by speaker and Chaput.

Fact: $H^*(G/P, \mathbb{Z}) \supset \oplus_{w\ non-\Lambda-minis}\, \mathbb{Z}\,\sigma^w = I_{min}$ is an ideal. Therefore $H^*_{min}(G/P, \mathbb{Z}) = H^*(G/P, \mathbb{Z})/I_{min} = \oplus_{w \Lambda-min}\, \mathbb{Z}\,\sigma^w$ has an algebra structure. Product is defined by: $\sigma^u \odot \sigma^v = \sum_{w\ \Lambda-min} t_{uv}^w \sigma^w$.

Proposition 2. *This defines an associative and commutative algebraic structure on $H^*_{min}(G/P, \mathbb{Z})$.*

Need to prove that if $\gamma^1, \cdots, \gamma^r$ are generators of $H^*(G/P, \mathbb{Z})$, then $\gamma^i \cup \sigma = \gamma^i \odot \sigma$ for all i and σ.

Proposition 3. *If h the hyperplane class, then $h \cup \sigma = h \odot \sigma$.*

To make this work together, need a bit more:

Definition 8. *Take w a Λ-miniscule element. Then $H(w)$ is slant-irreducible if any $x \in H(w)$ such that $\{y \in H(w) | c(y) = c(x)\} = \{x\}$ is maximal in $H(w)$.*

What is color, $c(x)$? Recall $H(w) = [1, \ldots, \ell]$, $w = s_{i_1}, \cdots s_{i_\ell}$, and c takes this to S, k mapping to α_{i_k}.

Fact: Any heap can be decomposed into slant-irreducible heaps.

Definition 9. *A heap $H(w)$ is called slant-finite if the Dynkin diagram of all slant-irreducible components of $H(w)$ are Dynkin diagrams of finite groups.*

Example. Look at pictures again (picture 5).

Theorem 5. *The conjecture is true for u, v, w Λ-miniscule and slant-finite.*

We prove this by induction on size of the heap. Each time you have a non-slant-irreducible heap you can decompose it, and Stembridge-Proctor gave some results on this.

Proposition 4. *Let $x \in H(u)$ such that $\{y \in H(u) | c(y) = c(x)\} = \{x\}$. Define v such that $H(v) = \{y \leq x\}$. If $\sigma^v \cup \sigma = \sigma^v \odot \sigma$, then $\sigma^w \cup \sigma = \sigma^w \odot \sigma$ for all $w \geq v$.*

In cases outside of slant-finite irreducible, were not able to prove equality. Lack of Poincare duality and other things also hamper general proof.

Can prove this in E_6/P_1. Contained in work of Thomas and Yong, but want to explain easy proof. The longest element of W^P is Λ-miniscule. See picture (6) at end of file. $H^*(E_6/P_1)$ is generated by two classes γ^1, γ^4 in degrees 1 and 4. $\gamma^1 = h$. Just need to prove $\gamma^4 \cup \sigma = \gamma^4 \odot \sigma$. But need even less: by Chevalley formula, have that if

- $1 \leq d \leq 3$ then $\dim H^d = 1$
- $4 \leq d \leq 7$ then $\dim H^d = 2$
- $d = 8$ then $\dim H^d = 3$
- $9 \leq d \leq 12$ then $\dim H^d = 2$
- $13 \leq d \leq 16$ then $\dim H^d = 1$

Use Lefschetz theorem here. Implies $\deg(\sigma^w) \geq 10$, and $\gamma^4 \cup \sigma = \gamma^4 \odot \sigma$ for all $\deg \sigma \leq 5$. Then need to do case of $\deg 8$.

Q: What about non-maximal parabolic subgroup P? It will correspond to more than one simple root (smallest elements in heap) and is covered by maximal parabolic case by taking products.

Please send corrections to taipale at math.umn.edu

Perrin

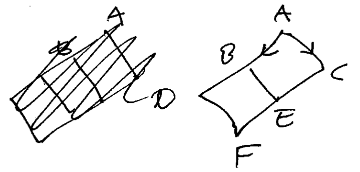
$$\overset{T}{\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}} \subset \overset{B}{\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}} \subset \overset{P}{\left(\begin{array}{c|c} u & u \\ \hline 0 & u \end{array}\right)} \subset G$$

picture $P = \{A, B, C, D, E, F\}$

$A \geq B, C$

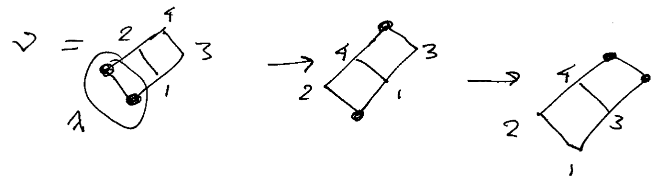
$B \geq D, E \geq F$

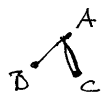
$C \geq E$



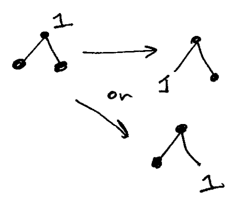
$\boxed{\text{jdt}}$

$\lambda =$



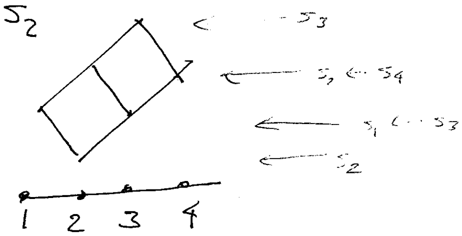


$A \geq B, C.$



$\boxed{\text{jdt not well-defined.}}$

$w = s_3 s_4 s_2 s_3 s_1 s_2$



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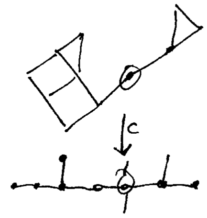
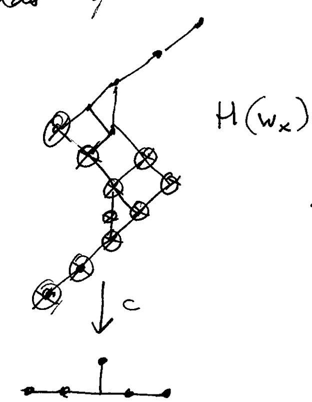


heap of a Kac Moody algebra.

This is a slant-finite heap.

Last pic 6



all ⊗ have to be contained in H(w) if $c^w_{uv} \neq t^w_{uv}$.

~~if you consider~~

3/23 N. Perrin "Toward a LR rule in the Kac-Moody setting"
(w/ P.E. Chaput)

① Notation: G = Kac-Moody gp
$T \subseteq B \subseteq P \subseteq G$; assume (for talk) P maximal.

$\left(\text{ex: } \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \right)$

W = Weyl gp of G, W_P = Weyl gp of P.

$X = G/P$ is an ind-variety;

consider $H^*(G/P; \mathbb{Z})$.

$$G/P = \coprod_{w \in W/W_P} BwP/P.$$

$\overline{BwP/P} =: X(w)$, $[X(w)] \in \sigma_w \in H_*(G/P, \mathbb{Z})$

$\{\sigma_w\}_{w \in W/W_P}$ basis for $H_*(G/P, \mathbb{Z})$

$\{\sigma^w\}_{w \in W/W_P}$ dual basis for $H^*(G/P, \mathbb{Z})$

Write $\sigma^u \cup \sigma^v = \sum_w c_{uv}^w \sigma^w$, c_{uv}^w = "LR coeff."

Problem: Give a combinatorial formula for c_{uv}^w.

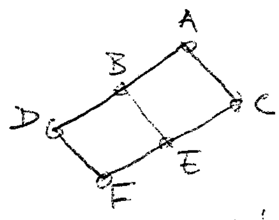
② Jeu de taquin (jdt)

P = poset, bounded below, $x \in P$, $\#\{y \leq x\} < \infty$.

Defn: • Order ideal in P: $I \subset P$ s.t. $\forall x \in I$, $y \leq x \Rightarrow y \in I$.

• Standard tableau T (ν, λ) pair of order ideals, $\lambda \subset \nu$
and order-preserving bijection
~~$\nu \setminus \lambda \to [1, \#$~~ $(\nu \setminus \lambda) \longrightarrow [1, \#(\nu \setminus \lambda)]$.

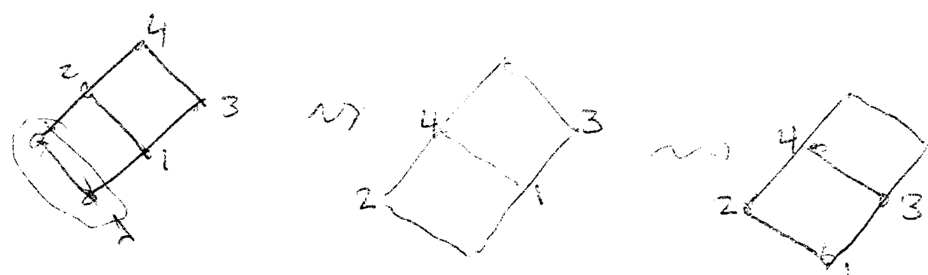
Ex: $P = \{A, B, C, D, E, F\}$, $A \geq B, C$; $B \geq D, E \geq F$, $C \geq E$



$\text{jdt}(T) = T'$, where $\text{shape}(T') = (*, \phi)$

- jdt slide: choose an elt $x \in \lambda$,
take the elt. $y \in \nu$ that covers x, with minimal label.
- do jdt slides until there's no vtx with empty label covered by a labeled vtx.

Ex:



Defn: A poset has the jdt property if jdt is well-defined,
i.e., indep't of choices of y's.

③ Λ-(co)miniscule elts

$W/W_P \longleftrightarrow W^P \subset W$, minimal length coset representatives.

P = max'l parabolic, Λ = associated weight

minimal embedding of $G/P \hookrightarrow \mathbb{P}(V(\Lambda))$, $V(\Lambda)$ = corresp. rep'n.

Def: An element $w \in W$ is $\underline{\Lambda\text{-miniscule}}$ if $\exists\ w = s_{i_1} \cdots s_{i_\ell}$ reduced,

s_i = reflection in α_i,

with $s_{i_k} \cdots s_{i_\ell}(\Lambda) = s_{i_{k+1}} \cdots s_{i_\ell}(\Lambda) - \alpha_{i_k}$)

i.e., $|\langle s_{i_{k+1}} \cdots s_{i_\ell}(\Lambda), \alpha_{i_k} \rangle| = 1$.

Cominiscule := miniscule for $\Lambda^\vee$.

Fact: $\{\Lambda\text{-miniscule elts}\} \subseteq W^P$.

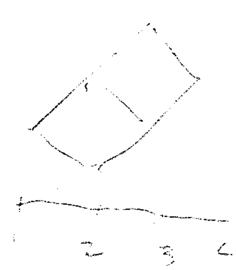
Def: G/P is $\underline{\text{miniscule}}$ if $\{\Lambda\text{-miniscule elts}\} = W^P$.

Def: Take $\tilde{w}$ reduced expr. for $w = s_{i_1} \cdots s_{i_\ell}$.

The heap of $\tilde{w}$ is $H(\tilde{w})$ = set $[1, \ell]$, with order:

p smaller than q if $p > q$ and s_{i_p}, s_{i_q} do not commute.

Ex: $\tilde{w} = s_3 s_4 s_2 s_3 s_1 s_2$



<u>Thm</u> (Stembridge, Proctor).

If w is Λ-minuscule, then

① $H(\tilde{w})$ does not depend on reduced word for w.

② $[e,w] \longrightarrow \{$order ideals in $H(\tilde{w})\}$ } is an order-preserving
$u \longmapsto H(u)$ } bijection.

③ $H(w)$ has the jdt property.

<u>Cor</u>:

For $u, v \leq w$ Λ-minuscule elts, for U a fixed std tableau of shape $(H(v), \phi)$, the number of std tableaux T with shape $(H(w), H(u))$ s.t. $jdt(T) = U$ does not depend on U.

Let t_{uv}^{w} be this number.

<u>Conj</u>: If u, v, w are Λ-minuscule, then $c_{uv}^{w} = t_{uv}^{w}$.

<u>Thm</u> (Thomas-Yong): The conj is true for X = minuscule.

<u>Thm</u> (Buch?-P.): The conj. is true for G/P, with G = f. dim'l alg. gp.

Applications:

Def: G/P is adjoint if isom. to closed orbit of G in $\mathbb{P}(\mathfrak{g})$.

Prop: For $\dim(G/P) = 2r+1$. If $\deg(\sigma^w) \leq r$, then w is Λ-cominuscule

Thm: $QH^*(G/P)$ is semi-simple iff G is not simply-laced.

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Rmks: • Know the conjecture is compatible w/ Chevalley formula
• " " " " compat. w/ recursion formulas. } From Thomas – Yong
• The t_{uv}^w's do define an alg. str. on $H^*(G/P)$. } new.

Consider ideal $\bigoplus_{w \text{ not } \Lambda\text{-minuscule}} \mathbb{Z}\cdot\sigma^w \subset H^*(G/P, \mathbb{Z})$.

Call it I_{min}.

Set $H^*_{min}(G/P, \mathbb{Z}) = H^*(G/P)/I_{min} \cong \bigoplus_{w = \text{minuscule}} \mathbb{Z}\cdot\sigma^w$.

Set $\sigma^u \odot \sigma^v = \sum_{w = \Lambda\text{-min}} t_{uv}^w \sigma^w$.

Prop: This defines an associative + commutative alg. str. on $H^*_{min}(G/P, \mathbb{Z})$.

[Q: Is there a relation w/ the Belkale-Kumar product?]

$\leadsto$ Need to prove: if $\{\gamma^1, \ldots, \gamma^r\}$ generate $H^*(G/P)$,
then $\gamma^i \cup \sigma = \gamma^i \odot \sigma$ $\forall i$, $\forall \sigma$.

Prop: If h = hyperplane class, $h \cup \sigma = h \odot \sigma$.

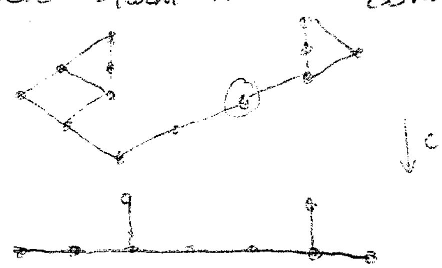
<u>Defn</u>: Take Λ-minuscule elt w. Then say $H(w)$ is <u>slant-irreducible</u> if any $x \in H(w)$ s.t. $\{y \in H(w) \mid c(y) = c(x)\} = \{x\}$ is max'l in $H(w)$.

(Here $c : H(w) = \{1, \dots, l\} \longrightarrow S = \{\alpha_i\}$ is a "coloring")
$$k \longmapsto \alpha_{i_k}$$

Any heap can be decomposed into slant-irreducible heaps.

<u>Defn</u>: $H(w)$ is <u>slant-finite</u> if the Dynkin diagram of each slant-irred. component is the diagram of a finite gp.

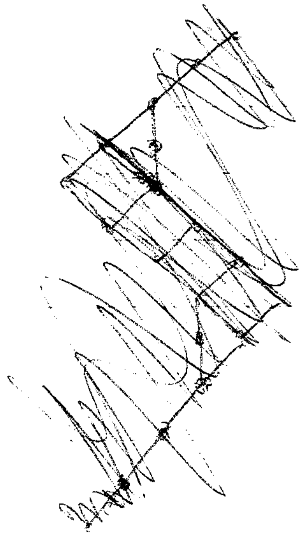
<u>Ex</u>:

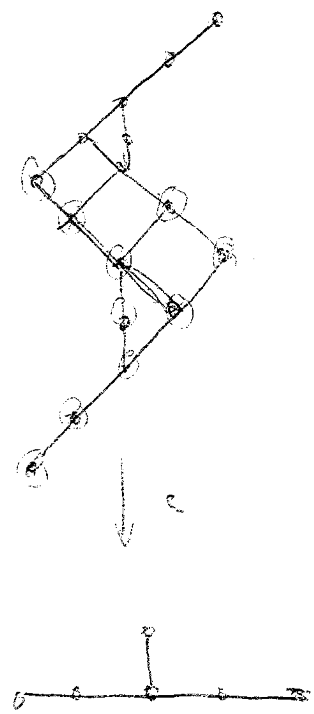


<u>Thm</u>: conj is true for u, v, w Λ-minuscule and slant-finite.

<u>Prop</u>: Let $x \in H(u)$ be s.t. $\{y \in H(u) \mid c(y) = c(x)\} = \{x\}$.
Define v s.t. $H(v) = \{y \leq x\}$.
Then if $\sigma^v \cup \sigma = \sigma^v \odot \sigma$, have
$$\sigma^w \cup \sigma = \sigma^w \odot \sigma \quad \text{for all } w \geq v.$$

Ex: E_6/P_1 (minuscule, already works in Thomas–Yong setup)
Longest elt in W^P is Δ-min.





← all circled elts must be in $H(w)$ if $c_{uv}^{w} \neq t_{uv}^{w}$.

$H^*(E_6/P_1)$ gen'd by two classes: γ^1, γ^4, of degrees 1, 4.

$\gamma^1 = h$ is hyperplane.

Have $\gamma^4 \cup \sigma = \gamma^4 \odot \sigma$.

$\dim H^{2d}$	1	2	3		1
d	$1 \leq d \leq 3$	$4 \leq d \leq 7$	8	$9 \leq d \leq 12$	$13 \leq d \leq 16$

(1)

MSRI 23.03.2009.

Towards a Littlewood Richardson rule for homogeneous spaces under Kac-Moody groups.

• Thank you and I thank the organisers for the opportunity to speak here. In this talk I want to report on joint work with P.-E. Chaput on combinatorial formulas to compute Littlewood-Richardson (LR) coefficients.

1) Notations.

• Let me first introduce some notation. I fix G a Kac-Moody (KM) group, and subgroups $T \subset B \subset P \subset G$ of G where T is a maximal torus, B a Borel subgroup and P a parabolic subgroup. I shall assume P maximal for simplicity of the exposition but all the results remain valid for any P.
I will denote by W the Weyl group of G and by W_P the Weyl group of P.

• The object I want to study is the homogeneous space $X = G/P$. In general (ie G KM) this is an ind-variety and especially its cohomology: $H^*(X, \mathbb{Q})$. One need to be careful because homology and cohomology are diff. for infinite dimensional varieties.

The Bruhat decomposition:

$$G/P = \coprod_{w \in W/W_P} BwP/P$$

is a cellular decomposition for G/P and therefore yields a basis for the homology:

the Schubert basis defined by $(\sigma_w)_{w \in W/W_P} = \left([\overline{BwP/P}]\right)_{w \in W/W_P}$. We define the Schubert basis of the cohomology as the dual basis $(\sigma^w)_{w \in W/W_P}$ in $H^*(X, \mathbb{Q})$. The LR-coefficients are the structure constants of the cup product in that basis:

$$\sigma^u \cup \sigma^v = \sum_{w \in W/W_P} c_{uv}^w \sigma^w.$$

Problem:

The most famous answer to this problem is the LR-rule using Jeu de Taquin jdT. This rule had been extended to any minuscule and cominuscule homogeneous space by H. Thomas and A. Yong very recently. Our result generalises these rules.

(2)

2) Jeu de Taquin:

So what is jdT. We start with P a poset which is bounded bellow ie $\forall x \in P$ the set $\{y \in P / x \geq y\}$ is finite.

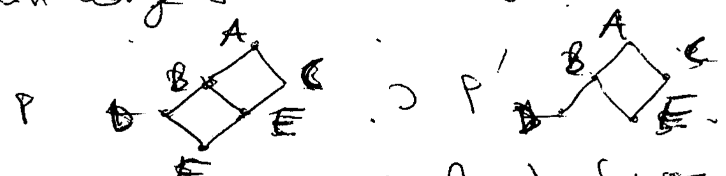
Def: An order ideal $I \subset P$ is a finite subset such that for $x \in I$ we have $(y \leq x \Rightarrow y \in I)$. We denote

$I(P) = \{$ order ideals in P $\}$.

- A standard tableau T in P is a pair $\lambda \subset \nu$ of order ideals and an increasing bijection $\nu \setminus \lambda \longrightarrow \{1, \dots, \#(\nu \setminus \lambda)\}$. The pair (ν, λ) is the shape of T.

Example: Let $P = \{A, B, C, D, E, F\}$ with order $A \geq B, C$
$B \geq D, E \geq F$
$C \geq E$

and $P' \subset P$ with $P' = \{1, 2, 3, 4, 5\}$
We represent the poset P as a graph (with vertices the elts of P) going from the biggest elt to the smallest with an edge for each covering relation:

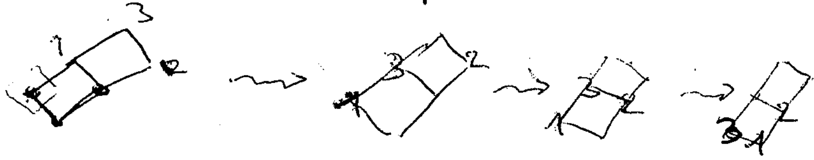


We define the order ideals $\lambda = \{D, E, F\}$ and $\nu = P$ of P and P'.
or $\lambda' = \{D, E\}$ or $\nu = P'$

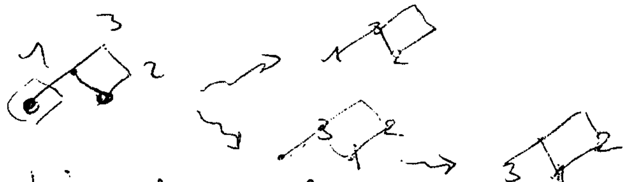
Jeu de Taquin is an inductive process where you do several jdT slides.
Def: A jdT slide is the following procedure: Take T a std tableau of shape (ν, λ).

- choose a vertex in λ (with no label or an empty vertex) and move the smallest label in ν (which is on the vertex say y) on x.
- The vertex y is left empty. Redo this until there is no empty vertex in ν covered by a label.

Let's do this on our examples: take the std tableaux:



here it is.



here jdT is not well defined

<u>Definition:</u> A poset has the jdT property if for any std tableau T the result of jdT(T) does not depend on the choice made in the rect° process (during the game).

(What does all this has to do with computing cohomology: there is a combinatorial framework were posets with jdT property ~~appear~~ appear as parametrising families of coho. classes.

3) <u>Λ-(co)minuscule elements:</u>

We have seen that cohomology classes are indexed by the set W/W_P -
In fact there are natural representatives of W/W_P in W given by:

$$W^P = \{\text{minimal length rep. of el}^{ts} \text{ in } W/W_P\} \text{ in each class}$$

$\varpi \in W/W_P$ there is a unique el^t w of minimal length in ϖ. Now we determine very special elements in W^P:

<u>Definition:</u> Let Λ be the fundamental weight associated to P (~~the~~ ~~G/P is~~ ~~embed~~) (the minimal embedding of G/P is $\mathbb{P}V(\Lambda)$ where $V(\Lambda)$ is the rep. of G of highest weight Λ)..

An element $w \in W$ is Λ-minuscule if $\exists$ a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ (where s_{i_k} is the simple reflection / simple root α_{i_k}) st:

$$\forall k,\ s_{i_k} \cdots s_{i_\ell}(\Lambda) = s_{i_{k+1}} \cdots s_{i_\ell}(\Lambda) - \alpha_{i_k}.$$

There is a dual notion of cominuscule elements for which we also obtain very similar results but for simplicity I shall skip this and concentrate on Λ-minuscule elements: <u>w is Λ-comin if it is min for $\Lambda^\vee = \sum a_i \varpi_i^\vee$ if $\Lambda = \sum a_i \varpi_i$.</u>

<u>Fact:</u> A Λ-minuscule element is in W^P.

<u>Def-Prop:</u> The variety $X = G/P$ is minuscule iff any element in W^P is Λ-minuscule.

~~[illegible]~~ For an element in W we introduce a poset $H(w)$, the heap of w:

[illegible]

Definition: let $w = s_{i_1} \cdots s_{i_\ell}$ ~~[illegible]~~ a reduced expression. The Heap of this expression is the poset $[1,\ell]$ with the ~~[illegible]~~ transitive closure of the ~~order~~: p is smaller than q if $p > q$ and $(s_{i_p} s_{i_q})^2 \neq 1$ i.e. s_{i_p} and s_{i_q} do not commute.

We have the following result:

Theorem (Stembridge and Proctor): w a Λ-minuscule.

Stemb. (I) ~~For w a Λ-minuscule element~~ $H(w)$ does only depend on w and not on the choice of a reduced expression.

(II) ~~For w a Λ-min. element~~, the map $[e,w] \to I(H(w))$, $u \longmapsto H(u)$ is an order preserving bijection between the Bruhat interval ($u \leq v \iff \overline{BuP/P} \subset \overline{BvP/P}$) and the set of order ideals in $H(w)$.

Proctor. (III) ~~For~~ $H(w)$ has the jdT property.

Corollary: For λ, μ, ν order ideals in $H(w)$, the number of tableaux of shape (ν/λ) which rectify on a std tableau U of shape μ does not depend on U.

We denote this number $t^{\nu}_{\lambda\mu}$.

Now we can state the:

Conjecture:
let $u, v \leq w$ be three Λ-minuscule elements. Then we have the equality:

$$c^{w}_{uv} = t^{H(w)}_{H(u),H(v)}$$

Remark that this is the LR rule for Grassmannians and that we have

Theorem (Thomas-Yong):
The conjecture is true for $X = G/P$ a minuscule homogeneous space.

Our main statement implies that:

Theorem (Chaput - P)

Before discussing the ideas of the proof let we describe some applications.

Definition: Let $X = G/P$, X is called an adjoint variety if ~~adjoint~~ X is the only closed orbit of G in $\mathbb{P}\mathfrak{g}$ where $\mathfrak{g} = \mathrm{Lie}(G)$ and ~~[illegible]~~ $\mathfrak{g}$ is considered with the adjoint action of G.

One can prove the following easy

Proposition: Let X be adjoint, then $\dim_{\mathbb{C}} X = 2r+1$ is odd and if $\sigma^w \in H^{2k}(X, \mathbb{Z})$ with $k \leq r$ then w is Nonminuscule.

In particular we are able to compute presentation and Giambelli formulas for $H^*(X, \mathbb{Z})$ for X adjoint and even for $QH^*(X, \mathbb{Z})$ the quantum cohomology. You can find a software for computing products in the q. Coh of these varieties on our webpages. We obtain

Thm (Chaput - P): X adjoint.

The $QH^*(X, \mathbb{Z})$ is semi-simple if and only if G is non simply laced. In that case there is an involution relating GW invariants for large degrees to GW invariants for low degrees.

4) Sketch of proof:

a) The strategy of the proof is very similar to that of Thomas and Yong: we prove that the numbers c_{uv}^w and $t_{uv}^w = t_{H(u)H(v)}^{H(w)}$ both satisfy various identities and that these identities together with few equalities $c_{uv}^w = t_{uv}^w$ imply the result. The identities are:

- c_{uv}^w and t_{uv}^w satisfy the Chevalley formula (see later).
- they satisfy recursion formulas: kind of induction formulas.
- the numbers t_{uv}^w define an associative & commutative algebra.

The first two points where already used by T.Y. the last one is new

Let me define the algebra structure: first we mod-out by the non (6)
Λ-minuscule elements:

Fact $J = \bigoplus_{\omega \text{ non } \Lambda\text{-min}} \mathbb{Z}\sigma^\omega$ is an ideal in $H^*(X,\mathbb{Z})$, therefore

$$H^*_{min}(X,\mathbb{Z}) = H^*(X,\mathbb{Z})/J$$ has an alg stru induced by the cup product $\overline{\sigma^u} \cup \overline{\sigma^v} = \overline{\sigma^u \cup \sigma^v}$.

Definition: on $H^*_{min}(X,\mathbb{Z})$ we define an algebra structure by:

$$\sigma^u \odot \sigma^v = \sum_{\omega\ \Lambda\text{-min}} t^w_{uv}\, \sigma^w.$$

If for T a std. tableau of shape (w,λ) we write $\sigma^T := \sigma^w$ then we have:

$$\sigma^u \odot \sigma^v = \sum_{\substack{T \text{ with} \\ \text{shape}(T)=(w,u) \\ T \rightsquigarrow U}} \sigma^T$$ where U is a std tableau of shape μ and $T \rightsquigarrow U$ means that T rectifies on U.

Proposition:

$\odot$ defines an associative and commutative algebra structure on $H^*_{min}(X,\mathbb{Z})$.

Proof: We only prove the associativity, the commutativity being easier (and already known). Take λ, μ, ν order ideals and fix U and V std tableaux of shapes μ and ν labelled by the indices $\{1,\dots,\#\mu\}$ and $\{\#\mu+1,\dots,\#\nu+\#\mu\}$.
We have

$$(\sigma^\lambda \odot \sigma^\mu) \odot \sigma^\nu = \Big(\sum_{U' \rightsquigarrow U} \sigma^{U'}\Big) \odot \sigma^\nu = \sum_{U' \rightsquigarrow U} \sum_{V'' \rightsquigarrow V} \sigma^{V''}$$

where shape$(U') = (\alpha,\lambda)$ and shape$(V'') = (\beta,\alpha)$. On the other hand we have

$$\sigma^\mu \odot \sigma^\nu = \sum_{V' \rightsquigarrow V} \sigma^{V'}$$ and because for any such V', $U \cup V'$ is a std tableau with shape$(V') = (\gamma, \mu)$ we have:

b) Slant products:

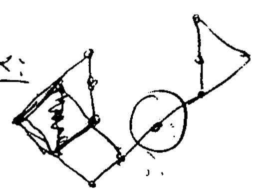
Together with this algebra we shall need the classification by Proctor of heaps of Λ-minuscule elements in terms of slant products.

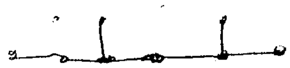
Def (i) H(u) is called slant irred if the only vertices $x \in H(u)$ st $\#\{y / c(x) = c(y)\} = 1$ are maximal vertices

(ii) the connected components of H(u) obtained by removing the non maximal vertices x with $\#\{y / c(x) = c(y)\} = 1$ are the slant irred components of H(u).

(iii) H(u) is called slant-finite if all its slant component are heaps of Λ-min elts for a finite dim group.

Thm (Chaput - P):
The conjecture is true for slant-finite Λ-minuscule elements.

ex:  is not slant-irred.
but is slant finite



Proposition: let $x \in H(u)$ st $\#\{y / c(y) = c(x)\} = 1$ (u Λ-min)
then define $H(v(x)) = \{y \leq x\}$.

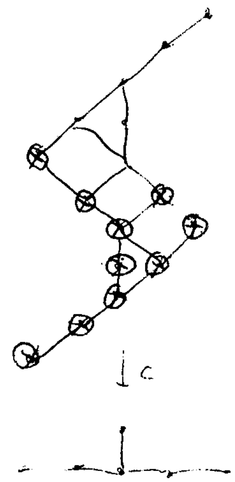
If $\sigma^u \odot \sigma^{v(x)} = \sigma^u \cup \sigma^{v(x)}$ then ~~this~~ $\geq v(x)$

$\sigma^u \odot \sigma^v = \sigma^u \cup \sigma^{v\xi}$

This more or less enables us to restrict ourselves the slant irreducible heaps. These were classified and we need to deal with a finite number of families. We end by case by case analysis and induction on the size of the heap.

Let me finish with an example of such a proof for E_6/P_1:

The heap of the longest elt (which is minuscule: E_6/P_1 is minuscule) is as follows:



The cohomology for E_6/P_1 is generated by two elements, one of degree 1 and the other one of degree 4 say γ^1 and γ^4.

We need to prove that $\gamma^i \cup \sigma = \gamma^i \odot \sigma$ for all i and all σ.

For $i=1$ we have $\gamma^1 = h$ and this is the compatibility with Chevalley formula. We need to prove it for γ^4.

Now in our proof we proceed by induction on the slant irreducible heaps so we know that $c_{uv}^w = t_{uv}^w$ for w non irreducible and also for w whose support is a Dynkin diagram of smaller type. This implies that w has to contain: i.e. $\deg w \geq 10$. This implies that for $\deg(\sigma) \leq 5$ we have $\deg(\gamma^4 \cup \sigma) \leq 9$ and the equality $\gamma^4 \cup \sigma = \gamma^4 \odot \sigma$.

Now the dim. of the pieces of coho acc. to the degree are:

	$d \in [1,3]$	$d \in [4,7]$	$d \in 8$	$d \in [9,12]$	$d \in [13,16]$
$\dim H^{2d}$	1	2	3	2	1

By Lefschetz the mult° by h is injective up to 8 and surjective after 8. ~~[illegible]~~
By the dim. it is also surj from 5 to 7. So by equality in degree 5 we are done up to degree 7. And if we prove degree 8 we are done using

$h \cup \sigma = h \odot \sigma$.

In degree 8 there are 3 classes: σ^8, γ^8 two of them contain / and by induction the result follows for these classes.

Now $\langle \sigma^8, \gamma^8, \mathrm{Im}\, h \rangle =$ all degree 8 classes so the result follows.