1 Ezra Miller

Title: Equivariant transversality and K-theoretic positivity

Keywords: equivariant K-theory, Kleiman transversality, homological transversality, Borel mixing space, flag variety

Summary: The lecture introduces equivariant K-theory to this audience, and then proceeds to sketch proofs of conjectures of Graham-Kumar and Griffeth-Ram about "positivity" for structure constants for torus-equivariant K-theory of flag varieties G/B. Included is a discussion of "positivity" for K-theory, which really means a certain alternation in signs.

Joint work with Dave Anderson and Steve Griffeth.

X smooth compact variety over complex numbers. Let $T = (\mathbb{C}^*)^n$. A vector bundle $E \to X$ is equivariant if T acts on the whole picture (on both E and X).

Equivariant K-theory: $K_T(X) = \oplus_E \mathbb{Z}[E]_T / \langle E = E' + E'' | 0 \to E' \to E \to E'' \to 0\rangle$. sum over all isomorphism classes of equivariant vector bundles. As written, it's a group; but it's also a ring, with multiplication being tensor products.

First example: $K_T(pt) = R(T) \cong \mathbb{Z}[\Lambda] = \oplus_{\lambda\in\Lambda} e^\lambda = \mathbb{Z}[e^{\pm\alpha_1}, \ldots, e^{\pm\alpha_m}$. Why? For $X = pt$, have $E = \oplus$ one-dim'l T-repns. (Notation: Λ weight lattice, αs weights.)

We will use Poincare duality: $\mathcal{F}$ coherent and equivariant sheaf on X, and $\mathcal{F}$ has resolution by vector bundles:

$$0 \leftarrow \mathcal{F} \leftarrow \mathcal{E}_0 \leftarrow \ldots \leftarrow \mathcal{E}_\ell \leftarrow 0.$$

(These are also equivariant morphisms of coherent sheaves.) Implies $[\mathcal{F}]_T := \sum_i (-1)^i [\mathcal{E}_i]_T$. In particular, we'll be interested in case where F is structure sheaf of subvariety (even Schubert variety).

If we had $\mathcal{F}$ and $\mathcal{G}$ and wanted class of products, we'd have to resolve both and do a lot of work. Turns out that $[\mathcal{F}]_T \cdot [\mathcal{G}]_T = \sum_i (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_T$. This alternation will come often in this talk: positivity in equivariant K-theory is alternating signs.

Q: hypotheses? A: smooth is the key hypothesis.

Setup for G/P and G/B: G a complex semi-simple algebraic group. Example to keep in mind is just SL_{n+1}. All theorems today are new for SL_{n+1}. $X = G/B$ (yes, B) where B is upper-triangular matrices. (Can think of P if you like: notation is easier with B.) $SL_{n+1}/B = Fl_{n+1}$ flag variety.

$T \subseteq B$ is a maximal torus. (If you take a subtorus, need positivity hypothesis; might touch on this at very end of talk.) In our example, T is just diagonal matrices in SL_{n+1}.

Cell decomposition: $G/B = \coprod_{w\in W} C^w$ (opposite Schubert cells) and $\overline{C^w} = X^w$ are opp Schubert varieties. So define *K-theory classes* $\mathcal{O}^w := [O_{X^w}]_T$. Given this, $K_T(X) = \oplus_{w\in W} R(T) \cdot \mathcal{O}^w$. Free as a module over $R(T)$ with rank equal to order of Weyl group.

Why should $K_T(X)$ be an algebra over the representation ring? Apply functoriality, looking at $X \to pt$.

So I've given a basis. What would the dual look like?

Q: When do we get finitely generated? A: rationality; different classes for every point in an elliptic curve, for example.

Ok, dual under what? We have equivariant euler characteristic $\chi_T : K_T(X) \to R(T)$ taking $[\mathcal{F}]_T \mapsto \sum_i (-1)^i [H^i(X, \mathcal{F})]$ (remember, $X = G/B$.) Torus acts on sections of sheaf, and so on H^is, so they're torus representations . Have $\chi_T(\xi^w \cdot \mathcal{O}_v) = \delta_v^w \in R(T)$: Kronecker delta. What is ξ^w? $\xi^w = [\mathcal{O}_{X^w}(-\partial X^w)]_T$, where $\partial X^w = \cup_{u>w} X^u$. Ideal sheaf of boundary divisor inside structure sheaf of X^w.

(Notice super/subscripts: Borel orbits and opposite Borel orbits, and they're transversal. Doesn't matter which is which, just matters that they're transverse!)

These ξs are dual in equivariant K-theory: in cohomology, don't care about what happens at boundary of things. You'd just say superscripts dual to subscripts. Here we have to worry about lower-dimensional stuff. Q: is this related to twisting by canonical class? A: Yes, but would have to look up how.

Ok, have equivariant K-theory of G/B with four bases: structure sheaves of Schubert (and opp) varieties, and duals of those two (the ξ-bases). What happens when we multiply and expand?

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum_{w \in W} c_{uv}^w \, \mathcal{O}^w$$

Have $c_{uv}^w \in R(T)$, so this is a Laurent polynomial in the e^αs. Same for ξs:

$$\xi^u \cdot \xi^v = \sum_{w \in W} p_{uv}^w \xi^w$$

with $p_{uv}^w \in R(T)$. Schubert calc is special and we want positivity for these coefficients! But what is positivity in this equivariant K-theory setting? (Q: how do we get LR coeff? A: need to take lowest-degree terms to get back to cohomology.)

Conjecture 1 (Griffeth-Ram)**.** $(-1)^{\ell(w)-\ell(u)-\ell(v) c_{uv}^w \geq 0}$

Conjecture 2 (Graham-Kumar)**.** *Same thing but with* p_{uv}^w.

This is K-theoretic positivity, whatever that's supposed to mean! (We will tell you what it means.)

Let $\alpha_1, \ldots, \alpha_n$ be the positive simple roots. (In SL_{n+1} case, have basis $e_1 - e_2, \ldots, e_n - e_{n+1}$.) Intuition: $1 - e^{-\alpha_i}$ should be positive. We'll get geometric explanation as part of the proof. Griffeths and Ram conjectured this from numerical evidence. Then for $J = (j_1, \ldots, j_n)$, using multi-index notation, $(1 - e^{-\alpha})^J$ "should" have a sign $(-1)^{|J|}$ where $|J| = \sum_i j_i$. I can rewrite this: $(e^{-\alpha} - 1)^J$ should have sign $+1$. *There is a sign error: best fixed in private!*

Definition 1 (Griffeth-Ram)**.** $c \geq 0$ *in* $R(T)$ *if* $c \in \mathbb{N}[e^{-\alpha_1} - 1, \ldots, e^{-\alpha_n} - 1]$.

Theorem 1 (Anderson-Griffeth-M)**.** *The Griffeth-Ram and Graham-Kumar conjectures are correct.*

Q: A priori there's not an implication one way or the other? A: Correct. To get from one to the other need to do Mobius inversion – lots of signs – no clear way to relate them. These are really different positivity results! The second one involves boundary divisors in definition, and other in proof. Q: are $e^\alpha - 1$ virtual zero-dim'l representations? A: Yes.

A bit of history: if you look at positivity in ordinary cohomology $H^*(X)$ then general statement for G/B was proved by Kleiman in 1974. One-line idea: $c_{uv}^w = \#(gX^u \cap X^v \cap X_w)$, where you translate X^u by generic g in group to get transverse intersection, which thus must be positive.

In ordinary K-theory this was proved by Brion (2001) shortly after Buch conjectured it. Buch noticed predictable sign alternation.

In $H_T^*(X)$ this was proved by Graham in 2001.

In $QH_T^*(X)$ this was proved by Mihalcea in 2006.

Now two missing: QK and QK_T. This would involve similar ideas but on moduli spaces.

1.1 Sketch of Proof

Borel: if you want to study something equivariant on X, you should study something ordinary on $(X \times ET)/T$ where $ET = (\mathbb{C}^\infty \setminus 0)^n$ (Borel mixing space). The action of T on X is not free, but action on ET is free and ET is equivariantly contractible. Can't do this in algebraic geometry directly, but Totaro and then Edidin-Graham (late 1990s) proved that $ET \sim (\mathbb{C}^m \setminus 0)^n = \tilde{ET}$. This is an approximation: choose m big enough that we can do what we want.

Set $\mathcal{X} = (X \times \tilde{ET})/T$. This maps to $\mathbb{P} := \tilde{B}T = \tilde{ET}/t = \mathbb{P}^{n-1} \times \cdots \mathbb{P}^{n-1}$ with map π.

Finally: $1 - e^{-\alpha_i}$ is the class $[\mathcal{O}_{H_i}]$, $H_i \subseteq i^{th}$ $\mathbb{P}^{n-1}$. This is the first step in the proof: exhibit these monomial classes geometrically. (a)

Next step (b) : express class that we want (which is c_{uv}^w) geometrically. Notice $\chi_T : K_T(X) \to K_T(pt)$. This gives $K(\mathcal{X}) \to K(\mathbb{P})$, $\mathcal{F} \mapsto \sum_i (-1)^i R^i \pi_* \mathcal{F}$.

Then for $Y \subseteq X$ get $[\mathcal{O}_Y]_T \in K_T(X)$, which gives rise to $\mathcal{Y} \subseteq \mathcal{X}$ giving $[\mathcal{O}_{\mathcal{Y}}^i] \in K(\mathcal{X})$.

Then $c_{uv}^w = \chi_T(\mathcal{O}^u\, \mathcal{O}^v\, \xi_w)$. For m sufficiently large, this equals $\chi_{\mathbb{P}}([\mathcal{O}_{\mathcal{X}^u}][\mathcal{O}_{\mathcal{X}^v}][\mathcal{O}_{\mathcal{X}_w(-\partial\mathcal{X}_w)}])$. This we need to calculate and it ought to be positive.

(c) Pick off coefficients of $(1 - e^{-\alpha})^J$. Restrict to J, then $\mathbb{P}_J$ (choose appropriately) on every line written in part (b).

(d) Look back at Kleiman's theorem and (b). Last two classes are transverse but first one isn't. Want to translate $[\mathcal{O}_{\mathcal{X}^u}]|_{\mathbb{P}_J}$ appropriately. (i) Use Dave Anderson's group action, but it's not transitive. Looking for homological transversality. (ii) Use Susan Sierra's 2007 results to conclude that higher direct functors vanish.

We can calculate this as an euler characteristic, but we need that it has only one term in it to conclude positivity. To get that, use Kawamata-Viehweg vanishing and an argument due to Brion. $\chi_{\mathbb{P}}(***) = \sum_i (-1)^i stuff$ where in stuff there is at most one non-zero term $i = \ell(w) - \ell(u) - \ell(v) + |J|$.

For Graham-Kumar, prove for any positive torus $S \subseteq T$ and $Y \subseteq X$ S-stable with rational singularities, get $[\mathcal{O}_Y]_T = \sum_{w \in W} c_w \, \mathcal{O}^w$ with $c_w > 0$ in $R(S)$.

Q: Which results does this imply? A: It implies everything for equivariant or non-equivariant K theory and cohomology.

Q: Brion used a boundary divisor too in his proof? A: Yes, but a different boundary divisor.

3/24 E. Miller "Equivariant transversality and K-theoretic ~~t~~ positivity"
(w/ D. Anderson + S. Griffeth)

X = smth compact var. / $\mathbb{C}$
$T = (\mathbb{C}^*)^n$
$\begin{array}{c} E \\ \downarrow \\ X \end{array}$ equivariant VB, so $T \circlearrowright E$.

I. Equivariant K-theory: $K_T(X) = \bigoplus_E \mathbb{Z}\cdot[E]_T \Big/ \langle E = E' + E'' \mid 0 \to E' \to E \to E'' \to 0 \rangle$

ring w/ $\otimes$ multiplication.

Ex: $K_T(pt) = R(T) \cong \mathbb{Z}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Z}\cdot e^{\lambda} = \mathbb{Z}[e^{\pm\alpha_1}, \dots, e^{\pm\alpha_n}]$

weight lattice

Pf: $X = pt \Rightarrow E = T\text{-rep'n} = \bigoplus (\text{1-dim'l } T\text{-rep'ns})$

"Poincaré duality": $\mathcal{F}$ = coherent sheaf/X. (T-equivariant)
equivariant resolution $0 \leftarrow \mathcal{F} \leftarrow \mathcal{E}_0 \leftarrow \mathcal{E}_1 \leftarrow \cdots \leftarrow \mathcal{E}_q \leftarrow 0$

$\Rightarrow [\mathcal{F}]_T = \sum_i (-1)^i [\mathcal{E}_i]_T$

To multiply coh. sheaves:
$[\mathcal{F}]_T \cdot [\mathcal{G}]_T = \sum_i (-1)^i [\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_T$

II. G = complex semisimple alg. gp (e.g. SL_{n+1})

$X = G/B$ (e.g. $B = \begin{bmatrix} & * \\ 0 & \end{bmatrix} \subset SL_{n+1}$) ($X = Fl_{n+1}$)

(Rmk: G/P also works, but notationally more complicated)

$T \subseteq B$ max'l torus $\left(T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right)$

$$G/B = \coprod_{w \in W} C^w, \qquad \overline{C^w} = X^w \text{ (opposite) Schubert variety.}$$

Set $\mathcal{O}^w := [\mathcal{O}_{X^w}]_T$. $\Longrightarrow K_T(X) = \bigoplus_{w \in W} R(T) \cdot \mathcal{O}^w$.

(In general, $K_T(X)$ is an algebra over $R(T) = K_T(pt)$, by pullback.)

Duality: equivariant euler char:

$$\chi_T : K_T(X) \to R(T)$$
$$[\mathcal{F}]_T \mapsto \sum_i (-1)^i [\underbrace{H^i(X, \mathcal{F})}_{T\text{-rep'n}}]$$

$\chi_T(\xi^w \cdot \mathcal{O}_v) = \delta_v^w \in R(T)$,

where $\mathcal{O}_v = [\mathcal{O}_{X_v}]_T$, $\xi^w = [\underbrace{\mathcal{O}_{X^w}(-\underbrace{\partial X^w}_{= \bigcup_{u > w} X^u})}_{\text{ideal sheaf in } \mathcal{O}_{X^w}}]_T$

Schubert calculus:

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum_{w \in W} c_{uv}^w \mathcal{O}^w, \qquad c_{uv}^w \in R(T).$$

$$\xi^u \cdot \xi^v = \sum_w p_{uv}^w \xi^w, \qquad p_{uv}^w \in R(T)$$

Conjecture (Griffeth-Ram 2004): $(-1)^{\ell(w)-\ell(u)-\ell(v)} c_{uv}^{w} \geq 0$ ("≥ 0" to be defined)

(Graham-Kumar 2007): $(-1)^{\ell(w)-\ell(u)-\ell(v)} p_{uv}^{w} \geq 0$.

Let $\alpha_1, \ldots, \alpha_n$ be simple roots (for SL_{n+1}, $e_1-e_2, \ldots, e_n-e_{n+1}$)

- $1-e^{-\alpha_i}$ "should" be ≥ 0
- For $J=(j_1,\ldots,j_k)$, $(1-e^{-\alpha})^J$ "should" have sign $(-1)^{|J|}$ $\boxed{\text{fix}}$
  $(e^{-\alpha}-1)^J$

Def (Griffeth-Ram): $c \geq 0$ in $R(T)$ if $c \in \mathbb{N}[e^{-\alpha_1}-1, \ldots, e^{-\alpha_n}-1]$.

Thm (AGM 2008): The conjectures of G-R and G-K are true.

(Rmk: No easy way to get from one conj. to the other.)

III. (History) $H^*(X)$: Kleiman 1974, $c_{uv}^{w} = \#(gX^u \cap X^v \cap X_w)$ (generic $g \in G$)

$K(X)$: Brion 2001 (conj. by Buch)

$H_T^*(X)$: Graham 2001

$QH_T^*(X)$: Mihalcea 2006

$QK_T(X)$? ($QK(X)$?)

IV. Proof sketch:

Borel: equivariant on X $\longleftrightarrow$ ordinary on $(X \times ET)/T$,
$ET = (\mathbb{C}^\infty \setminus 0)^n$

Totaro, Edidin-Graham (late 1990's) $ET \approx (\mathbb{C}^m \setminus 0)^n =: \tilde{E}T$ $(m \gg 0)$

$$\begin{array}{l} \mathcal{X} = (X \times \tilde{E}T)/T \\ \downarrow \pi \\ \mathbb{P} = \tilde{B}T = \tilde{E}T/T = \mathbb{P}^{m-1} \times \cdots \times \mathbb{P}^{m-1} . \end{array}$$

$1 - e^{-x_i} = [\mathcal{O}_{H_i}]$, $H_i \subseteq i^{\text{th}}\ \mathbb{P}^{m-1}$ hyperplane.

$\Rightarrow$ "monomial" classes are exhibited geometrically. (a)

(b) Express c_{uv}^w geometrically.

$\chi_T : K_T(X) \to K_T(pt) \rightsquigarrow K(\mathcal{X}) \to K(\mathbb{P})$

$\mathcal{F} \longmapsto \sum_i (-1)^i R^i \pi_* \mathcal{F}$

$Y \subseteq X \rightsquigarrow [\mathcal{O}_Y]_T \rightsquigarrow \mathcal{Y} \subseteq \mathcal{X}$
$\rightsquigarrow [\mathcal{O}_{\mathcal{Y}}] \in K(\mathcal{X})$.

$$c_{uv}^w = \chi_T(\mathcal{O}^u \cdot \mathcal{O}^v \cdot \xi_w) \underset{(m \gg 0)}{=} \chi_{\mathbb{P}}([\mathcal{O}_{\mathcal{X}^u}] \cdot [\mathcal{O}_{\mathcal{X}^v}] \cdot [\mathcal{O}_{\mathcal{X}_w}(-\partial \mathcal{X}_w)])$$

(c) Pick off coeff. of $(1 - e^{-x})^J$.
(Restrict everything to $\mathbb{P}_J = \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_n}$)

(d) Translate $\mathfrak{X}^{u}|_{\mathbb{P}_J}$ "generically".

(i) Anderson's gp action (~~not~~ transitive)

(ii) Sierra's homological transversality (2007)

$\rightsquigarrow$ higher Tor's vanish

(e) Vanishing thm:

$$\chi_{\mathbb{P}}(\cdots) = \sum_i (-1)^i \,(\text{stuff})$$

at most one nonzero term, appearing in $i = \ell(w) - \ell(u) - \ell(v) + |J|$.

(Argument similar to Brion's, using K-V vanishing.)

For G-K conj.:

$S \subseteq T$ "positive" subtorus,

$Y \subseteq X$ S-stable, rat'l singularities.

$$\Rightarrow [\mathcal{O}_Y]_T = \sum_{w \in W} c_w \mathcal{O}^v \;, \text{ with } c_w \geq 0 \text{ in } R(S).$$

joint w/ Dave Anderson
Steve Griffeth

$\boxed{\text{Equivariant transversality and K-theoretic positivity}}$

MSRI Workshop: Combinatorial, Enumerative and Toric Algebraic Geometry
24 March 2009, Tuesday 9:30-10:30

(1)

I. service: K-theory

II. combinatorial equivariant positivity; statement of thm.

III. history

IV. proof sketch: transversality on mixing spaces

X = smooth compact variety$/\mathbb{C}$

$T = (\mathbb{C}^*)^n$

vector bundle $T \circlearrowright \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ equivariant if $T \circlearrowright$

$$K_T(X) = \bigoplus_E \mathbb{Z}\cdot[E]_T \Big/ \langle E = E' + E'' \mid 0 \to E' \to E \to E'' \to 0 \rangle$$

ring with mult.: $\otimes$

e.g. $K_T(pt.) = R(T) \cong \mathbb{Z}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{Z}\cdot e^\lambda = \mathbb{Z}[e^{\pm\alpha_1}, \ldots, e^{\pm\alpha_n}]$.

weight lattice

pf: $X = pt. \Rightarrow \underset{\text{T-rep}}{E} = \bigoplus$ 1-dim. T-reps.

Poincaré duality: $\mathcal{F}$ coherent$/X$ $\Rightarrow$ $0 \leftarrow \mathcal{F} \leftarrow \mathcal{E}_0 \leftarrow \mathcal{E}_1 \leftarrow \cdots \leftarrow \mathcal{E}_\ell \leftarrow 0$

$$\Rightarrow [\mathcal{F}]_T = \sum_i (-1)^i [\mathcal{E}_i]_T$$

$$[\mathcal{F}]_T \cdot [\mathcal{G}]_T = \sum_i (-1)^i [\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_T$$

G complex semisimple alg. gp. ~~for SL_{n+1}~~ it's all still new, and ~~just as hard~~ no harder or easier

$X = G/B$ $\qquad SL_{n+1}/B = Fl_{n+1}$ $\qquad B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

$= \bigcup_{w \in W} X^w$ $\qquad T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

Everything works for G/P, too, but with more annoying notation; if you know what "minimal length coset representatives" are, then you probably don't need me to tell you how to generalize the statements — but feel free to ask. For the benefit of everyone else, I'll stick to G/B, which you should think of as Fl_n.

T = max. torus in B

$\mathcal{O}^w := [\mathcal{O}_{X^w}]_T \Rightarrow K_T(X) = \bigoplus_{w \in W} R(T)\cdot\mathcal{O}^w$ ← free over $R(T)$ of rank $|W|$, via $\begin{smallmatrix} X \\ \downarrow \\ pt. \end{smallmatrix}$ $\begin{smallmatrix} K_T(X) \\ \uparrow \\ K_T(pt.) \end{smallmatrix}$

(equivariant vector bundles pull back)

dual basis?? duality?

$$\chi_T : K_T(X) \to R(T)$$

$$[\mathcal{F}]_T \mapsto \sum_i (-1)^i \underbrace{H^i(X,\mathcal{F})}_{T\text{-rep.}}$$

"virtual sum of T-reps"

$$\chi_T(\xi^w \cdot \mathcal{O}_v) = \delta^w_v \in R(T)$$

for $\xi^w = [\mathcal{O}_{X^w}(-\partial X^w)]_T$, $\quad \partial X^w = \bigcup_{u>w} X^u$

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum_{w \in W} c^w_{uv}\, \mathcal{O}^w \qquad c^w_{uv},\ p^w_{uv} \in R(T).$$

$$\xi^u \cdot \xi^v = \sum_{w \in W} p^w_{uv}\, \xi^w$$

Griffeth-Ram conj (2004): $(-1)^{\ell(w)-\ell(u)-\ell(v)}\, c^w_{uv}$ "≥ 0"

Graham-Kumar conj (2007): p^w_{uv}

$\alpha_1, \dots, \alpha_n$ simple roots $\qquad SL_{n+1}: e_1 - e_2, \dots, e_n - e_{n+1}$

- $1 - e^{-\alpha_i}$ should be ≥ 0
- $J = (j_1, \dots, j_n) \Rightarrow (1-e^{-\alpha})^J$ has sign $(-1)^{|J|}$
  $\Rightarrow (e^{-\alpha} - 1)$ $\qquad +1$

<u>Def</u> [GR] $c \geq 0$ in $R(T)$ if $c \in \mathbb{N}[e^{-\alpha_1}-1, \dots, e^{-\alpha_n}-1]$.

<u>Thm</u> [Anderson-Griffeth-M 2008]: [GR] + [GK] ✓.

<u>History</u>

H^*	Kleiman 1974	$c^w_{uv} = \#(gX^u \cap X^v \cap X_w)$ generic $g \in G$
K	Brion 2001 (conj: Buch)	
H^*_T	Graham 2001	
K_T	AGM	
QH^*_T	Mihalcea 2006	
QK_T	?	

<u>Proof sketch</u> Borel: equivar. on X $\leftrightarrow$ ordinary on $(X \times ET)/T$

$ET = (\mathbb{C}^\infty \setminus 0)^n$

(a) exhibit "monomial" classes geometrically

Totaro, Edidin, Graham: late 1990s $ET \approx (\mathbb{C}^m \setminus 0)^n =: \tilde{E}T$

$$\mathcal{X} := (X \times^T \tilde{E}T)/T$$

$$\pi \downarrow$$

$$\mathbb{P} := \tilde{B}T = \tilde{E}T/T = \mathbb{P}^{m-1} \times \cdots \times \mathbb{P}^{m-1}$$

moral: $\mathbb{P}$ = macaulay [illegible], geometric [illegible] equivariant point [illegible] = of T stack

$$1 - e^{-\alpha_i} = [\mathcal{O}_{H_i}] \qquad H_i \subseteq i^{th}\ \mathbb{P}^{m-1}$$

(b) express the class we want geometrically on $\mathbb{P}$

$\chi_T : K_T(X) \to K_T(pt) \rightsquigarrow \chi_{\mathbb{P}} : K(\mathcal{X}) \to K(\mathbb{P})$ = trivial group

$Y \subseteq X \rightsquigarrow [\mathcal{O}_Y]_T \in K_T(X) \quad \rightsquigarrow \quad \mathcal{F} \overset{\pi}{\mapsto} \sum_i (-1)^i R^i \pi_* \mathcal{F}$

$$\mathcal{Y} \subseteq \mathcal{X} \rightsquigarrow [\mathcal{O}_{\mathcal{Y}}]$$

$$c_{uv}^{w,J} = \chi_T(\mathcal{O}^{u,J} \mathcal{O}^v \xi_w) \underset{m \gg 0}{=} \chi_{\mathbb{P}}\left([\mathcal{O}_{\mathcal{X}^u|_{\mathbb{P}_J}}] \underbrace{[\mathcal{O}_{\mathcal{X}^v}] \cdot [\mathcal{O}_{\mathcal{X}_w(-\partial \mathcal{X}_w)}]}_{}\right)$$

(c) pick off coeff. on $(1-e^{-\alpha})^J$

^{J}, ^{J}, $|_{\mathbb{P}_J}$ $\quad [\mathbb{P}_J] = [\mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_n}]$ dual to $[H_1]^{j_1} \cdots [H_n]^{j_n}$

(d) translate $\mathcal{X}^u|_{\mathbb{P}_J}$ generically!

(i) Anderson's group action on $\mathcal{X}$ (not transitive!)

(ii) Sierra's homological transversality Thm [2007]

$[\underbrace{\ \ }$: $\mathcal{X}^v \pitchfork \mathcal{X}_w(-\partial) \Rightarrow$ need $\mathcal{X}^u|_{\mathbb{P}_J} \pitchfork$ in H_T^*, this is enough; but here, need:

(e) show $\chi_{\mathbb{P}}(\cdots) = \sum$ at most 1 term

$\chi_{\mathbb{P}}(\cdots) = \sum_i (-1)^i (?)$ based on [Brion]: Kawamata-Viehweg + hard, detailed calculations to verify hypotheses

$\Rightarrow$ nonzero term has $i = \ell(w) - \ell(u) - \ell(v) + |J|$. $\square$

More direct analog of Brion's argument $\Rightarrow$ equivariant Brion: this was Brion's rephrasing of ordinary K-theoretic positivity

<u>Thm</u> [AGM; GrKu conj]: $S \subseteq T$ "positive", $Y \subseteq X$ S-stable w/ rat. sings

$$\Rightarrow [Y]_T = \sum_{w \in W} c_w \mathcal{O}^w \text{ with } c_w \geq 0 \text{ in } R(S).$$

see [illegible]'s talk for [illegible]