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Title: Convex bodies, semi-groups of integral points, algebras of finite type, and geometry of linear series on varieties

Keywords: convex bodies, mixed volumes, affine varieties, intersection theory of divisors, Newton convex set

Summary: Elementary proofs in combinatorics imply interesting results in algebraic geometry. Asymptotic behaviour of semigroups of integral points can be related to asymptotic results on graded algebras of finite type over a field k, and also to results on big divisors in algebraic geometry. Several approximation theorems are discussed.

Elementary proofs in combinatorics imply some interesting results in algebraic geometry. All follow from observations on semigroups of integral points. All joint with AG Khovanskii; there are three preprints on the arXiv. One is expository. A new one is to appear next week with same title as talk.

Work is very related to that of Lazarsfeld and Mustata's in "Convex bodies associated to linear series." Also motivated by results of Okounkov.

Plan of talk:

- semigroups of integral points in $\mathbb{Z}^n$
- $\mathbb{Z}^n$-valuation on a field F, giving results on Hilbert functions of graded subalgebras of polynomials in $F[t]$ (of almost finite type)
- Applications to algebraic geometry and convex geometry.

F field over k, finitely generated of finite transcendence degree n. Valuation $v : F\backslash\{0\} \to \mathbb{Z}^n$. For algebraic geometry, take $F = \mathbb{C}(X)$ where X is an n-dimensional irreducible variety. (That's it – not necessarily complete or smooth!)

Start with semigroup $S \subset \mathbb{Z}^n$. Associate to this $Con(S)$, the cone of S, $L(S)$ the span of S in $\mathbb{R}^n$, and $G(S) \subset \mathbb{Z}^n$ group generated by S.

Theorem 1 (Approximation theorem)**.** *$Con(S) \cap G(S) =: Reg(S)$. S approximates $Reg(S)$ "very well."*

We know: if we have 2\$ bills and 5\$ bills, $2, 5 \in \mathbb{Z}$, for large $n \in \mathbb{Z}$, n can be paid using these 2\$ and 5\$ bills. (Real-life application!)

Theorem 2 (Khovanskii)**.** *There exists $C > 0$ such that for finitely generated S, $Con(S)$ and $x \in Con(S) \cap G(S)$, if distance of x from ∂S is $\geq C$, then $x \in S$.*

The approximation theorem is a generalization of this theorem. Rewrite:

Theorem 3 (Approximation, version 2)**.** *S any semi-group $\subset \mathbb{Z}^n$. Take any strongly convex cone $Con \subset Con(S)$. Then there exists a constant $C > 0$ depending on choice of cone such that if $dist(0, x) > C$ and $x \in Con \cap G(S)$, then $x \in S$.*

Example: draw graph of $y = \sqrt{x}$ in $\mathbb{R}^2$. Look at $S \subset \mathbb{Z}^2$ above this curve. Notice that $Con(S)$ is the upper half-plane. If we take a strongly convex cone Con in the upper half-plane based at the origin, we have always some extra, but for large distance from the origin all points in Con are also in S.

Take a *non-negative semigroup* $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$, $S \nsubseteq \mathbb{Z}^n \times \{0\}$. Associate to this a Hilbert function. S_k is all elements of S at level k: for $\pi : \mathbb{Z}^m \times \mathbb{Z} \to \mathbb{Z}$, and $\pi_S : S \to \mathbb{Z}$, $S_k = \pi_S^{-1}(k)$. Then the Hilbert function is $H_S(k) = \#S_k$.

A bit more notation: $\pi(G(S)) \subset \mathbb{Z}$ is a semigroup, and $m = m(S)$ is the index of this in $\mathbb{Z}$. (Project on the vertical axis, and notice that this subsemigroup eventually is all multiples of $m(S)$.) $ind(S)$ is the order of $G(S) \cap \mathbb{Z}^n$ in $L(S) \cap \mathbb{Z}^n$. These are invariants. For large values of k have only non-zero parts of Hilbert function when k is multiple of m.

Motivated by definition of Newton polytope,

Definition 1. *The Newton convex set of S is $Con(S) \cap \pi^{-1}(m)$. This lives in $\mathbb{R}^n \times \{m\}$. We call the Newton convex set $\Delta(S)$.*

The volume of this newton convex set is supposed to be responsible for asymptotic behavior of Hilbert function.

Example: Given finite set of points $x_1, \dots, x_n \in \mathbb{Z}^n$, consider $\tilde{x_i} = (x_i, 1)$ and S semigroup generated by $\tilde{x_i}$, $\Delta(S)$ is the convex hull of the $\tilde{x_i}$s. This plays an important role in toric geometry.

Theorem 4 (Main theorem)**.** *S non-negative semigroup in $\mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$. Then*

$$\lim_{k \to \infty} \frac{H_S(mk)}{k^q} = \frac{Vol_q(\Delta(S))}{ind(S)}$$

where $q = \dim_{\mathbb{R}} \Delta(S) = \dim L(S)$.

This follows from the approximation theorem.

Q: do we assume index finite? A: Yes.

For $S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$, $Con(S)$ strongly convex, look at $\hat{S}_p$ the subsemigroup generated by S_p, semigroup at level p.

Theorem 5. *Let $p >> 0$, and $a_p = \lim_{\ell \to \infty} \frac{H_{\hat{S}_p}(p\ell)}{\ell^p}$, and $q = L(L(S))$. Then*

$$\lim_{k \to \infty} \frac{a_{km}}{k^q} = \lim_{k \to \infty} \frac{H_S(mk)}{k^q}.$$

Lazarsfeld and Mustata proved a similar statement for the case where $S \subset \mathbb{Z}^n_{\geq 0}$ and used it to prove the Fujita approximation theorem for big divisors on projective algebraic varieties.

Now we have all the results we need to look at applications to algebra and algebraic geometry.

F a finitely generated extension of k, of transcendence degree n. Take valuation $v : F \backslash \{0\} \to \mathbb{Z}^n$ (ordered). Such a valuation always exists. Usual definition of valuation has $v(fg) = v(f) + v(g)$, and $v(f+g) \geq \max(v(f), v(g))$. We add one more condition: $v(f) = v(g)$ implies $\exists\lambda$ such that $v(f - \lambda g) > v(f)$. Also require v is onto.

Usual example of a valuation is this: let $f(x) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$

Extension for power series: $v_t : F[t] \to \mathbb{Z}^{n+1}$. $f(t) = \sum_{k=\ell}^m a_k t^k$ and $v_t(f) = (v(a_\ell), \ell)$.

Let $B \subset F[t]$ a graded subalgebra, $L \subset F$ finite dim.

Theorem 6. *For B of almost finite type, $S(B) = v_t(B) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$ is a strongly convex non-negative semigroup.*

Then $H_B(k) = \dim B_k$, $B = \oplus_{k\geq 0} B_k$, have

$$\lim_{k\to\infty} \frac{H_B(k)}{k^q} = \frac{Vol(\Delta(S(B)))}{\mathrm{ind}(S(B))}.$$

Similar result for Fujita type.

Applications to algebraic geometry come from $F = \mathbb{C}(X)$ and X an n-dimensional irreducible algebraic variety. For D a divisor on X, $Ł(D) = \{(f) + D > 0\}$. For X complete this is finite-dimensional. Then $R(D) = \oplus_{k\geq 0} Ł(kD)t^k$ is of almost finite type.

From this we can recover many results in algebraic geometry that were known for big divisors and asymptotics of spaces of line bundles, for instance.

Q: Can you talk about choice of valuations? A: Need valuation from $\mathbb{C}(x) \to \mathbb{Z}^n$ that is onto and has one-dimensional ** (did not hear). Favorite comes from taking smooth point of X and coordinate system around that smooth point, and consider the valuation constructed on the polynomials in that system. Okounkov etc. consider valuations coming from flags, so have some non-smooth points, but with resolution of singularities can transform to this nice case.

Finally want to list applications to algebras of almost finite type:

- for a divisor D, $vol(D) = vol_n(\Delta(R(D)))$
- $\lim_{k\to\infty,\, m|k} \frac{\dim(Ł(kD))}{k^q}$ exists and is equal to $Vol_q(\Delta(R(D)))$
- generalization of Fujita approximation
- Kushnirenko theorem
- Hodge inequality on surfaces (Hodge index theorem). Get a birational version.
- Alexander-Fenchel inequality for mixed volumes.

Also constructed birational intersection theory of divisors on X.

3/24 K. Kaveh, "Convex bodies, semigps, ..."
(w/ A.G. Khovanskii) (on arXiv)

– cf. Lazarsfeld–Mustata, "Convex bodies assoc. to linear series"

Plan:
- semigps $\subset \mathbb{Z}^n$
- $\mathbb{Z}^n$-valuations on a field F $\quad \nu : F \setminus \{0\} \longrightarrow \mathbb{Z}^n$

$\rightsquigarrow$ Hilbert fns of graded subalg. of $F[t]$
(of "almost finite type") $\quad$ (F/k f.g., tr.deg $= n$)
- Applications to alg. geom, convex geom.
($F = \mathbb{C}(X)$, $X = n$-dim'l cx variety)

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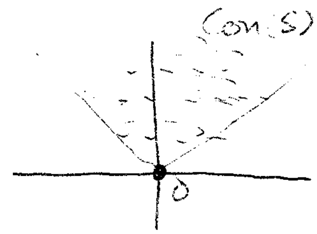
$S \subset \mathbb{Z}^n$ semigp.
- $\mathrm{Con}(S)$ = cone gen'd by S



- $L(S)$ = linear span of $S \subseteq \mathbb{R}^n$
- $G(S) \subseteq \mathbb{Z}^n$ gp gen'd by S.

Approximation thm: $\mathrm{Con}(S) \cap G(S) =: \mathrm{Reg}(S)$.

S approximates $\mathrm{Reg}(S)$ "very well"

Problem: \$2 coins, \$5 bills; ~~[illegible]~~ $2, 5 \in \mathbb{Z}$;
for large n, n can be paid in multiples of \$2, \$5

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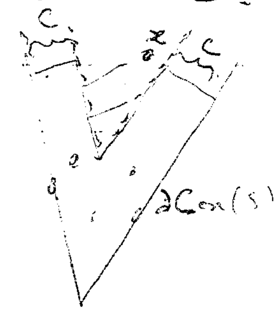
f.g semigp S.

Thm (Khovanskii)

$\exists$ $C > 0$ s.t. for $x \in Con(S) \cap G(S)$,
if distance of x from ∂S (→ $\partial Con(S)$) is $\geq C$, then $x \in S$.



Approx. thm

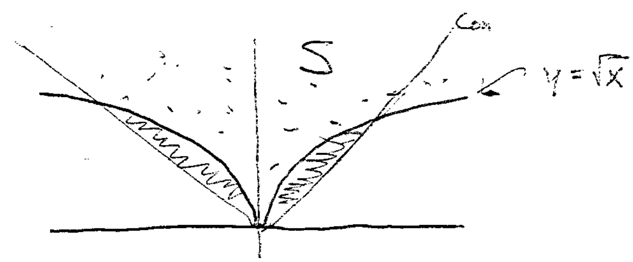
$S \subseteq \mathbb{Z}^n$ any semigp.

$Con \subset Con(S)$ s.t. :

- Con is strongly convex, $x \in Con \cap G(S)$
- $Con \cap \partial Con(S) = \{0\}$

Thm $\exists$ $C > 0$, s.t. $dist(x, 0) > C \Rightarrow x \in S$.

Ex. $S \subset \mathbb{Z}^2$



$Con(S)$ = upper half plane

$S \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$, $S \not\subset \mathbb{Z}^n \times \{0\}$ ("non-negative semigp")

$S_k = \pi_S^{-1}(k)$, for $\pi : \mathbb{Z}^n \times \mathbb{Z} \to \mathbb{Z}$ projection, $\pi_S = \pi|_S$.

Hilbert fn: $H_S(k) = \# S_k$.

Let $m(S)$ = index of $\pi(G(S)) \subseteq \mathbb{Z}$.

$ind(S)$ = index of $G(S) \cap \mathbb{Z}^n$ in $\mathbb{Z}^n \cap L(S)$ (so finite)

(For large k, $H_S(k) \neq 0 \Leftrightarrow k$ = multiple of $m = m(S)$.)

Let $\mathbb{B}$ = almost f. type.

<u>Thm</u>: $S(\mathbb{B}) := \nu_t(\mathbb{B}) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geq 0}$

is a strongly cvx non-negative semigp.

<u>Cor</u>: $H_{\mathbb{B}}(k) = \dim \mathbb{B}_k$, $\mathbb{B} = \bigoplus_{k\geq 0} \mathbb{B}_k$

$$\rightsquigarrow \lim_{k\to\infty} \frac{H_{\mathbb{B}}(k)}{k^q} = \frac{\mathrm{Vol}(\Delta(S(\mathbb{B})))}{\mathrm{ind}(S(\mathbb{B}))}.$$

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<u>Applications to alg. geom</u>

$F = \mathbb{C}(X)$, X = n-dim'l alg. var., complete

D = divisor on X, $\mathcal{L}(D) = \{(f) + D \geq 0\}$ (f. dim'l)

$R(D) = \bigoplus_{k\geq 0} \mathcal{L}(kD) t^k$ is of almost f. type.

Choosing valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ (as above),
recover some things about asymptotics of this line bundle/divisor...

(Ex: v coming from choice of coord. system at smth. pt of X.)

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<u>Applying results on algs of almost f. type</u>:

- for D = div. $\mathrm{Vol}(D) = \mathrm{vol}_n(\Delta(R(D)))$
- for $m = m(R(D))$, $\lim_{k\to\infty} \frac{\dim(\mathcal{L}(kD))}{k^q}$ exists, $= \mathrm{Vol}_q(\Delta(R(D)))$

- (slight) Generalization of Fujita approximation.
- Kushnirenko thm ⇝ Bernstein thm
- Hodge inequalities on surfaces (Hodge index thm)

[Replace intersection index for divisors with "intersection index" for subspaces...]

- Alex-Fenchel inequalities for mixed volumes.

[Rmk: These intersection indices need not be integers.]

Generalization of Kushnirenko thm

$L \subset \mathbb{C}(X)$

$$[L, \ldots, L] := \#\{x \in X \mid f_1(x) = \cdots = f_n(x) = 0, \; f_i \in L\}$$

(throwing out poles + common zeros)

Thm $\rightsquigarrow$ $$[L, \ldots, L] = \frac{n! \, \mathrm{Vol}(\Delta(B(L)))}{\mathrm{ind}(B(L))}$$