What can we learn about matroids from K-theory?

David E Speyer

Based on *The Tropical Grassmannian* with Bernd Sturmfels, *Tropical Linear Spaces*, *A Kleiman-Beritini Theorem for sheaf tensor products* with Ezra Miller, *A Matroid Invariant via the K-Theory of the Grassmannian* and further work in progress, some with Alex Fink.

How Algebraic Geometers can help combinatorialists: What are matroids? Why should they be related to Grassmannians? A specific program for finding constructions and theorems of interest to matroid theorists.

A specific problem I solved this way: The combinatorial problem that got me interested in this subject, and its solution

Some pretty geometry: A few nice reformulations of this solution – log chern classes, toric varieties.

Remaining Questions

We write $[n] = \{1, 2, \ldots, n\}$. A matroid of rank d on $[n]$ is a collection of d-element subsets of $[n]$, obeying certain axioms.

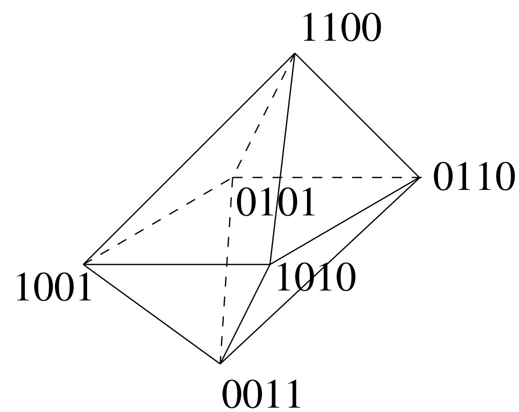
Motivating example: Take a $d \times n$ matrix $A = (v_1 \mid v_2 \mid \cdots \mid v_n)$ of rank d. The collection of $(i_1, \ldots, i_d) \in \binom{[n]}{d}$ such that v_{i_1}, v_{i_2}, $\ldots$, v_{i_d} is a basis for $\mathbb{C}^d$ is a matroid of rank d on $[n]$. Such a matroid is called realizable over $\mathbb{C}$.

In other words, for a point $x \in G(d, n)$, the set of I for which the Plücker coordinate $p_I(x)$ is nonzero forms a matroid.

Geometrically, a matroid describes the combinatorics of a hyperplane arrangement $v_1^\perp$, $v_2^\perp$, $\ldots$, $v_n^\perp$ in $\mathbb{P}^{d-1}$.

To any collection $M \subset \binom{[n]}{d}$, we assign the polytope

$$P_M := \text{ConvexHull } (e_I)_{I \in M} \text{ where } e_{i_1,\ldots i_d} := e_{i_1} + \cdots + e_{i_d} \in \mathbb{Z}^n.$$



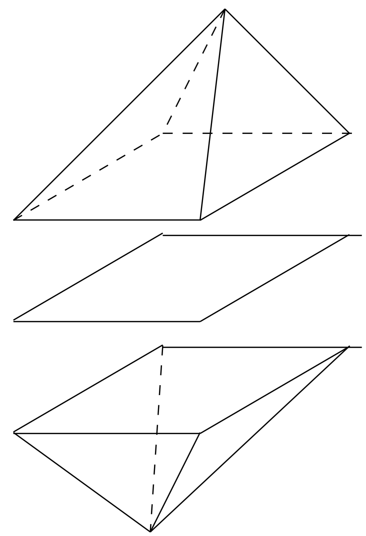
Proposition. *(Gelfand, Goresky, MacPherson, Serganova) M is a matroid if and only if $M \neq \emptyset$ and every edge of P_M is parallel to $e_i - e_j$ for some i, $j \in E$.*

We call such a P_M a *matroidal polytope.*

Motivation for considering P_M: Let M be the matroid associated to a point x in $G(d,n)$. Then $\overline{(\mathbb{C}^*)^n \cdot x}$ is a toric variety, with polytope P_M.

So the GGMS criterion says that a matroid "looks like a $G(d,n)$ torus orbit closure in dimension 1."

I am interested in decomposing matroidal polytopes into smaller matroidal polytopes. This combinatorial problem occurs when understanding Kapranov's Chow quotients of the Grassmannian; Hacking-Keel-Tevelev's compactification of the moduli space of hyperplane arrangements; and my tropicial linear spaces.



Many combinatorial invariants are linear under such decompositions. Let P_M be decomposed into matroidal polytopes and let P_F, $F \in \mathcal{F}$, be the interior faces of this decomposition.

Classical Matroid Combinatorics: The Tutte polynomial satisfies $t_M(z,w) = \sum_{F \in \mathcal{F}} (-1)^{\dim(P_M)-\dim(P_F)} t_F(z,w)$. (*Tropical Linear Spaces*, S., Section 6)

Polyhedral Geometry: The Erhart polynomial is defined by $\mathrm{Erhart}(P_M)(t) = \#(t \cdot P_M \cap \mathbb{Z}^n)$. Clearly, $\mathrm{Erhart}(P_M)(t) = \sum_{F \in \mathcal{F}} (-1)^{\dim(P_M)-\dim(P_F)} \mathrm{Erhart}(P_F)(t)$.

Combinatorial Hopf Algebras: Matroids form a "combinatorial Hopf algebra" so, by general theory, we get a quasi-symmetric function $Q(M)$ for each matroid M. This obeys $Q(M) = \sum_{F \in \mathcal{F}} (-1)^{\dim(P_M)-\dim(P_F)} Q(F)$. (*A quasisymmetric function for matroids*, L. Billera, N. Jia and V. Reiner)

A natural question:

Form the vector space V spanned by matroids of rank d on n elements, moduli relations for polytope decompositions. So our invariants are linear functionals on V. Does this vector space have a simple description?

Herm Derksen has an invariant, valued in $\mathbb{Q}^{\binom{n}{d}}$, and conjectures that the resulting map $V \to \mathbb{Q}^{\binom{n}{d}}$ is injective.

Alex Fink and I believe we have a proof of injectivity and we understand that Derksen does as well.

All of these examples should come from K-theory

What does this mean?

Let $x \in G(d,n)$ and let M be the corresponding matroid. Let $X = \overline{(\mathbb{C}^*)^n x}$. As mentioned above, this is a toric variety with polytope P_M.

Above, we considered the Erhart polynomial of P_M. This is simply $\dim H^0(X, \mathcal{O}(n))$. **Fact:** This depends only on M and n and adds in decompositions.

More generally, for any vector bundle V on $G(d,n)$, we can consider $\sum(-1)^i \dim H^i(X,V)$. **Fact:** This depends only on M and V and adds in decompositions.

K-theory – the more sophisticated way to understand this

Let Y be an algebraic variety, in our case, $G(d,n)$. $K_0(Y)$ is the abelian group generated by coherent sheaves on Y, subject to the relation $[A]+[C]=[B]$ whenever there is a short exact sequence $0 \to A \to B \to C \to 0$. **Key Fact:** $[\mathcal{O}_X]$ only depends on M and adds in decompositions.

For any vector bundle V, if $0 \to A \to B \to C \to 0$ is exact, so is $0 \to A \otimes V \to B \otimes V \to C \otimes V \to 0$. So $[V][A] := [V \otimes A]$ is a well defined element of $K_0(Y)$, and is linear in $[A]$. Thus, $[\mathcal{O}_X \otimes V]$ depends only on M and adds in decompositions.

Also, if Y is compact then $\chi : [E] \mapsto \sum (-1)^i \dim H^i(Y,E)$ is a well defined, linear, map from $K_0(Y) \to \mathbb{Z}$.

So that's why $\sum (-1)^i \dim H^i(X,V) = \chi([V][X])$ depends only on M and V, and adds in decompositions.

My hope: All of these combinatorial invariants should occur as $\chi([\mathcal{O}_X][E])$ for some $[E] \in K_0(G(d,n))$.

Evidence:

(1) GGMS show that being a matroid has to do with one dimensional torus orbits. The Goresky-MacPherson "moment graph" method shows that (equivariant) K-theory is determined by the one dimensional torus orbits in $G(d,n)$.

(2) $K_0(G(d,n))$ has dimension $\binom{n}{d}$; Derksen's universal invariant lives in a vector space of dimension $\binom{n}{d}$. (Although, in both cases, I expect that we can reduce to lower dimensional vector spaces.)

(3) For many matroid invariants, $F(M) = 0$ when dim X is too small. Like multiplying by a class of high codimension.

A combinatorial problem that calls for inventing a new matroid invariant

In *Tropical Linear Spaces*, I made the following conjecture. Let $\Delta(d,n) = \text{ConvexHull } (e_I)_{I \in \binom{E}{d}}$.

Conjecture. *If $\Delta(d,n)$ is divided into matroidal polytopes, then there are at most*

$$\frac{(n-c-1)!}{(d-c)!(n-d-c)!(c-1)!}$$

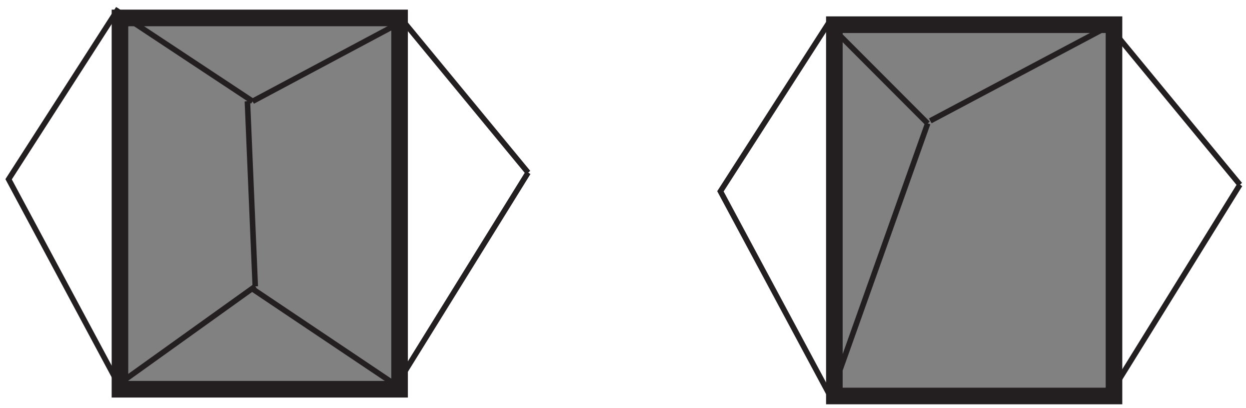
internal faces of dimension $n-c$.

I can now show:

Theorem. *The above holds when all the matroids involved are realizable over $\mathbb{C}$.*

I know when the bound is tight

Theorem. *The bound* $\frac{(n-c-1)!}{(d-c)!(n-d-c)!(c-1)!}$ *is achieved (for all* c*) whenever all of the matroids in our decomposition are direct sums of series-parallel matroids. Otherwise, we will have strict inequality at* $c = 1$ *(and possibly for other* c *as well.)*



This isn't just a result about $\Delta(d, n)$.

Theorem. *If* M *is any matroid and* P_M *has two decompositions into matroidal polytopes, in which all of the matroids used are direct sums of series-parallel matroids, then the two decompositions use the same number of internal faces of each dimension.*

To prove the equality claim, I had to construct an invariant of matroids which adds in valuations, and gives the same value on every series-parallel matroid polytope. (More precisely, we want $h_M(t) = t^c$ if M is a direct sum of c series-parallel matroids.) To prove the inequality, I needed the coefficients of $h_M(t)$ to alternate in sign.

Using K-theory, I can construct h_M for any matroid M.

I know how to show that the coefficients alternate, and hence prove the above conjecture, when M corresponds to a point of $G(d,n)(\mathbb{C})$. (Proof uses Kawamata-Viewheg vanishing.)

Not only does h_M have the properties above, but many others: it is invariant under series-parallel extension, multiplies in direct sums, and behaves well under many other standard matroid constructions.

Parallel extension is

$$(v_1 \mid v_2 \mid \cdots \mid v_i \mid \cdots \mid v_n) \to (v_1 \mid v_2 \mid \cdots \mid v_i \mid v_i \mid \cdots \mid v_n)$$

In the hyperplane arrangement picture, it inserts two copies of the same hyperplane.

Series extension is $\perp \circ$(Parallel Extension)$\circ \perp$.

A series-parallel matroid is a matroid obtainable from $\{\{1\},\{2\}\}$ by repeated series and parallel extensions. Except for some boundary cases, we could also say that a series-parallel matroid is one obtained from $\{\{1\},\{2\}\}$ by parallel extension and orthogonal complement.

Note that series-parallel hyperplane arrangements have no moduli.

So, what is h_M?

Let $F\ell(1, d, n-1; n)$ be the space of partial flags of dimension $(1, d, n-1)$. We have $p : F\ell(1, d, n-1; n) \to G(d, n)$ and $q : F\ell(1, d, n-1; n) \to \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$. It turns out that $q_* p^* [\mathcal{O}_X]$ is a polynomial in the class of a hypersurface of degree $(1, 1)$. The invariant h_M is this polynomial.

In particular, the lowest degree term of h_M measures the number of maximal faces when decomposing P_M. This is the classical β invariant; it can be seen in cohomology, without going to K-theory. We see that this is the degree of the correspondence $X \leftarrow F\ell(1, d, n-1; n) \to F\ell(1, n-1; n)$.

Several directions to go from here . . .

Positivity proof is Brion's positivity result applied to $F\ell(1, n-1; n)$. (Published version doesn't make this explicit.) Applying Brion's result directly on $G(d, n)$ only gives weaker statements.

In general, it looks like we are using the fact that X is $(\mathbb{C}^*)^n$ equivariant to strengthen Brion's results.

Can this be systemized? Will Ezra's talk help?

I said

> It turns out that $q_* p^* [\mathcal{O}_X]$ is a polynomial in the class of a hypersurface of degree $(1, 1)$.

Why? The current proof is an elegant, but unilluminating, combinatorial argument. Here is a suspicion about what the moral argument should be.

There is a correspondence

$$\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \to E \leftarrow \left(\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee\right) //(\mathbb{C}^*)^{n-1}.$$

Here the right hand side is the Chow quotient. (Related to Bernd Sturmfels' talk!)

The result is that $q_* p^* [\mathcal{O}_X]$ is a push-pull from the Chow quotient. Is there some theorem saying that a nice enough equivariant class is push-pulled from the Chow quotient?

Relation to log geometry Instead of push-pulling to $\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$, we can consider $X \cap p(q^{-1}(B))$, for various B in $\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$. In particular, if B is $\{\mathrm{pt}\} \times (\mathbb{P}^{n-1})^\vee$, then $X \cap p(q^{-1}(B))$ is a normal crossing compactification of the corresponding hyperplane arrangement. (Kapranov/Hacking-Keel-Tevelev).

Let (Y, D) be any such normal crossing compactification of the hyperplane arrangement complement. The coefficients of $h_M(t)$ are holomorphic Euler characteristics of the "Chern classes" of the logarithmic tangent bundle of (Y, D). In particular, they depend only on Y. This explains why h_M is unchanged by parallel extensions.

I suspect that h_M is just the first treasure in the K-theory goldmine. **Some questions for the future:**

Is knowing $[\mathcal{O}_X]$ equivalent to knowing Derksen's universal invariant? How do we express them in terms of each other?

What is the dimension of the vector space spanned by $[\mathcal{O}_X]$, as we range through all matroids of rank d on n elements? What is the polyhedral geometry of the cone they span?

What other specializations of K-classes might be useful to matroid theorists?

Matroids come from T orbit closures in $G(d, n)$. In general, we could take H orbit closures in any G/P. What positivity properties of these K-classes are waiting to be found, and what are their combinatorial meanings?

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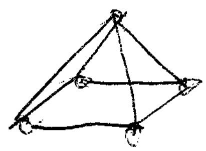
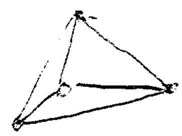
$d \times n$ mtx $\xrightarrow{\text{row span}}$ point in $G(d,n)$



$\{12, 13, 14, 23, 24\}$

→ is a matroid



$\{12, 14, 23, 24\}$

<u>not</u> a matroid

//

<u>Q</u>: What's known about dimension of "space of valuations"

<u>A</u>: ...

$$K_0(G(d,n)) = \bigoplus \mathbb{C}\cdot[\mathcal{O}_{X_\lambda}], \quad \lambda \subset d \times (n-d) \text{ box}.$$

X supported in dimension l $\Rightarrow$ $[\mathcal{O}_X] \in \bigoplus_{|\lambda| \le l} \mathbb{C}\cdot[\mathcal{O}_{X_\lambda}]$.

For toric varieties $X = \overline{T\cdot x}$, $l = n-1$.

<u>Rmk</u>: H. Derksen has a gen. fn.

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<u>Q</u>: Non-rep'ble matroids?

<u>A</u>: $T = (\mathbb{C}^*)^n$, X = T-inv't class/subvar in $G(d,n)$

$$K_T(G(d,n)) \longrightarrow \bigoplus_{p \in G(d,n)^T} K_T(p) = \bigoplus \mathbb{Z}[\Lambda]$$

image characterized by $f_{S_i} - f_{S_j} \equiv 0 \mod (u_i - u_j)$, $(f_I)_{I \in \binom{[n]}{d}}$.

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d] & \left((\mathbb{P}^{n-1}) \times (\mathbb{P}^{n-1})^\vee\right) /\!/ T \arrow[d] & \text{Chow quot.} \\ \mathbb{P}^{n-1}\times(\mathbb{P}^{n-1})^\vee \arrow[r, dashed] & \left(\mathbb{P}^{n-1}\times(\mathbb{P}^{n-1})^\vee\right)/_\alpha T & \text{GIT-quot} \end{tikzcd}$$

Q: Can you work directly w/ matroid polytopes (+ toric varieties), rather than through $G(d,n)$?

Brion's formula:

P

$$\sum_{\alpha\in P\cap\mathbb{Z}^n} e^\alpha = \sum_{v = \text{vtx of } P} h_{\underbrace{X_P}_{\text{toric var.}}}(e^\alpha)$$

$$\sum_{v\in P_M} \cdots$$

Q: Flavor of Derksen's invt?

A: Given M = matroid on $[n]$,
$$\chi_I(M) = \begin{cases} 1 & \text{if } I \text{ is lex first elt of } M, \\ 0 & \text{else} \end{cases}$$

$$\chi_{I,w}(M) = \begin{cases} 1 & \text{if } I \text{ is } w\text{-lex first} \\ 0 & \text{else} \end{cases}$$
$(w\in S_n)$

$\Longrightarrow \binom{n}{d}\times n!$ invts, project onto S_n-invariants.

Q: Tutte polyn comes from K-thy of matroids: i.e., obeys

$$t_M(z,w) = t_{M\setminus e}(z,w) + t_{M/e}(z,w)$$

$$\begin{array}{cc} \updownarrow & \updownarrow \\ X\cap G(d,n-1) & X\cap G(d-1,n-1) \end{array}$$

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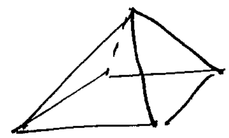
David Speyer, 24 March 2009

matroidal polytopes:

1

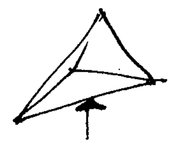
$\{12, 13, 14, 23, 24\}$



is still a matroid (all edges in direction $e_i - e_j$)

BUT $\{12, 14, 23, 24\}$

2



not in ok direction!

is NOT a matroid: some edge not in $e_i - e_j$ direction.

decomposition: have ~~constraints~~ on how finely we can chop up matroids. After all, example 2 shows we can't go too far. So need combinatorial invariants linear under decomp.

Note: looking at $\underline{H^i(X, \mathcal{O}(k))}$ which depends only on k, M (X toric var. assoc. to P_M).

Q: What do you $\underline{\text{know}}$ about space of valuations?

A: $K_0(G(d,n)) = \bigoplus \mathbb{C} \cdot \mathcal{O}_{X_\lambda}$ $\quad \lambda \subseteq d \times (n-d)$ box

For X supported in dimension ℓ, $[\mathcal{O}_X] \in \bigoplus_{|\lambda| \le \ell} \mathbb{C}[\mathcal{O}_{X_\lambda}]$

in our case, $\ell = n+1$. This is a bound on what you can get; ...

Derksen apparently knows bound now.

Q: How to deal with non-representable matroids?

A: ~~Using~~ $T = (\mathbb{C}^*)^n$. For X T-invariant class in $G(d,n)$...

$$① \; K_T(G(d,n)) \hookrightarrow \bigoplus_{\substack{\text{fixed} \\ \text{pts of} \\ G(d,n)}} K_T(pt)$$

Recall $K_T(pt) = \oplus \mathbb{Z}[\Lambda]$

Image: $f_{s_i} - f_{s_j} \equiv 0 \bmod u_i - u_j$ for $(f_I)_{I \in \binom{[n]}{d}}$

Compute Hilbert series, clear denominators, get polynomial which is class you want.

Q: Can you reformulate in terms of vector space V?

A: Yes; need to...

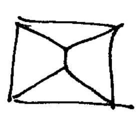
Q/A:

$$\begin{array}{ccc} G(d,n) & \hookrightarrow & \mathbb{P}^{\binom{n}{d}-1} \\ & \nwarrow_{\text{realizability}} & \uparrow \\ & & X \end{array}$$

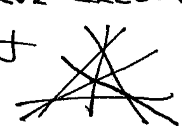
Plucker embedding

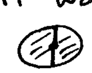
$\left[\begin{array}{l}\text{Get something ~~to~~ that } \underline{\text{acts}} \text{ like a toric} \\ \text{variety in } G(d,n) \text{ (K-theoretically).}\end{array}\right]$

Motivation: *how many series-parallel ~~pieces~~ pieces used to chop up a matroid* is an invariant! and the starting place for this work. *Where do you find invariants adding in valuations?* K-theory.

 $4t + \bullet - 5t^2 + 2t^3$

but many matroids don't have decomp:

hyperplane arrangement  doesn't;

work out h_M and get the decomp. it wants: 

X as torus orbit closure has dim $n-1$

$$\begin{tikzcd} \mathcal{F}\ell(1,d-1,n-1)^{2n-3} \arrow[r, two heads] \arrow[d, "+(d-1)+(n-d-1)"] & \mathcal{F}\ell(1,n-1)^{2n-3} \subset \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \\ X^{n-1} & \end{tikzcd}$$

dimension counting.

$$t_m(z,w) = \beta(m)(z+n)t \ldots$$
$$h_m(t) = \beta(m)t + \ldots$$

Positivity proof really runs Brion's proof on $\mathcal{F}\ell(1,n-1)$.

Q: How do you use torus action to control rat'l singularities on $\mathcal{F}\ell(1,n-1;n)$?

A: X^{n-1} is normal, has rational singularities, so its image upstairs has rat'l singularities. Then need extension of Brion's result to $Y \to G/P$ generically finite, and eventually use positivity of β-invariant.

Correction to diagram

$$\begin{tikzcd} \cancel{\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \leftarrow} & E \arrow[r] \arrow[d] & (\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee) /\!/ (\mathbb{C}^*)^{n-1} \arrow[d, dashed, "\text{Chow quotient}"] \\ & \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \arrow[r, dashed] & (\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee) /\!/_\alpha T \end{tikzcd}$$

Q: there are GIT quotients that fit. Why not think about those?

A: Think up top because Chow is natural and gives $q_* p^*[\mathcal{O}_X]$ is a polynomial in class of hypersurface of degree $(1,1)$. No natural GIT choice presents itself.

Q: Can you get invariants out of toric/matroid side and forget Grassmannian?

A: To understand whole class need to know how to pair vector bundles with toric variety. Can use Brion's formula: if you have polytope P, want to know $\sum_{\alpha \in P \cap \mathbb{Z}^n} e^\alpha$, it's $= \longrightarrow$

$$\sum_{\substack{v \text{ vertex} \\ \text{of } P}} h_{T_v(P)}(e^{\alpha})$$

GKM comp. of Euler char of line bdl.

Can do this for vector bundles:

$$\sum_{v \in P_M} h_{T_v(P)} \cdot (\mathrm{poly}(e^{\alpha}))$$

<u>Ex</u> for $(x_1 + \dots + x_d)$, $\sum_{I \subset M} h_{T_I(P_M)} \cdot \left(\sum_{i \in I}(x_i)\right)$.

Q: How does Derksen do this?

A: Given M matroid on $[n]$,

$$\chi_I(M) = \begin{cases} 1 & \text{if } I \text{ is lexicographically first element of } M \\ 0 & \text{else} \end{cases}$$

Make this not dependent on labelling:

$$\chi_{I,w}(M) = \begin{cases} 1 & w\text{-lexico} \\ 0 & \end{cases}$$

$w \in S_n$

Comment: Derksen's inv spec Tutte polyomial also has K-theoretic interpretation, though not vector bundle K-theory:

$$t_M(z,w) = t_{M \setminus e}(z,w) + t_{M/e}(z,w)$$

$$\uparrow \qquad\qquad \uparrow$$

$$X \cap G(d, n-1) \qquad X \cap G(d-1, n-1)$$

(restricting to smaller Grassmannians)

— Are these related?

Fix $\ell \subset H^{m-1}$, $\Omega = \{L : \ell \subset L \subset H\}$

$\beta(n) = \#(\Omega \cap X)$

deform; it breaks to

(This rescaling used when Hodge proved Schubert basis ...)