1 Diane Maclagan

Title: Tropical bounds on nef cones

Keywords: tropical geometry, toric variety, nef cone, Neron-Severi group,

Summary: In this talk recent results of the speaker and Angela Gibney are presented. In particular, upper and lower bounds on the nef cone are computed using tropical geometry. Examples considered throughout the talk are toric varieties and the moduli space of stable curves.

Joint with Angela Gibney.

Goal: Use techniques from tropical geometry to bound/compute the nef cone.

Y a projective variety, and $N^1(Y)$ is $\mathbb{Q}$-Cartier divisors up to numerical equivalence: $D\ D'$ if $D.C = D'.C$ for all curves $C \subset Y$. $N^1(Y)_\mathbb{R} = N^1(Y) \otimes \mathbb{R}$. $N_1(Y)$ is curves on Y up to numerical equivalence and $N_1(Y)_\mathbb{R} = N)1(Y) \otimes \mathbb{R}$.

$\overline{Eff}(Y) \subset N^1(Y)_\mathbb{R}$ is the closure of the cone spanned by effective divisors, and $\overline{NE}(Y) \subset N_1(Y)_\mathbb{R}$ is the closure of the cone spanned by effective curves.

The nef cone, then, is

$$nef(Y) = \{[D] \in N^1(Y)_\mathbb{R} | [D] \cdot [C] \geq 0 \forall C \in \overline{NE}(Y)\} \quad (1)$$
$$= \overline{NE}(Y)^\vee \quad (2)$$
$$= \overline{Amp(Y)} \quad (3)$$

Question: given Y, we'd like to compute $\overline{Eff}(Y)$, $\overline{NE}(Y)$, and $nef(Y)$. This is too hard, but in this talk we will give upper bound on cone of curves, lower bound of cone of curves, and discuss how tropical geometry contributes to these computations.

Example (1): $Y = X(\Sigma)$ projective toric variety, for instance $Y = Bl_p(\mathbb{P}^2)$, $N^1(Y)_\mathbb{R}$ is spanned by the torus-invariant divisors (orbit closures of codimension one torus-orbits).

Q: Are we assuming varieties are smooth? A: No.

These torus-invariant divisors don't form a basis for this vector space, but we do understand their relations! $\overline{Eff}(Y)$ is the cone generated by the torus-invariant divisors. Likewise, $N_1(Y)_\mathbb{R}$ is spanned by torus-invariant curves, and $\overline{NE}(Y)$ the cone generated by torus-invariant curves.

Look at *picture 1* at end of file.

Example (2): $\overline{M}_{o,n}$ moduli space of stable genus-zero curves with n marked points. It's a smooth variety of dimension $n-3$. This moduli space has a combinatorial stratification: the codimension k stratum is curves with k nodes. Curves with one node are our boundary divisors. Curves with $n-4$ nodes are dimension one strata: curves we call "F-curves." See *picture 2*.

$N^1(\overline{M}_{0,n})_\mathbb{R}$ is spanned by the boundary divisors. However, $\overline{Eff}(\overline{M}_{0,n})_\mathbb{R}$ is NOT contained in the cone generated by boundary divisors (worked out by Keel, others for $n \geq 6$). **Check**

$N_1(\overline{M}_{0,n})_\mathbb{R}$ is spanned by the F-curves, and

Conjecture 1 (F-conjecture)**.** *$\overline{NE}(\overline{M}_{0,n})$ is cone generated by F-curves.*

This is true for $n \leq 7$ by Keel and McKernan.

One last trivial observation: to know the class in $N_1(Y)_{\mathbb{R}}$ of $C \subseteq Y$ it suffices to know $C \cdot \delta_i$ where $\delta_1, \ldots, \delta_s$ span $N^1(Y)_{\mathbb{R}}$.

Now, a strategy for obtaining upper and lower bounds.

Y projective variety. Fix an embedding $i : Y^d \hookrightarrow X(\Delta)^n$, where $X(\Delta)$ is a toric variety that need not be complete or projective. Make it simplicial for today. Think about Δ being a 3-dimensional fan inside of 7-dimensional space, for instance. One more assumption: have embedding i, so assume

$$i^* : A_{n-1}(X(\Delta))_{\mathbb{R}} \to N^1(Y)_{\mathbb{R}}$$

is surjective. Also assume that toric variety has a projective completion.

Define $\delta_i := i^*(D_i)$.

Lemma 1. *If $C \subset Y$ and $C \cap T \neq \emptyset$, then $C \cdot \delta_i = w_i \geq 0$. Then $\sum w_i \underline{u}_i = 0$ where $\underline{u}_i$ is the first lattice point on the ith ray of Δ .*

These are essentially Fulton/Sturmfels Minkowski weights.

Definition 1. *A non-negative Minkowski weight on Δ is $W = (w_i) \in \mathbb{N}^{|\Delta(1)|}$ such that $\sum w_i \underline{u}_i = 0$.*

W gives rise to $[W] \in N_1(Y)_{\mathbb{R}}$ such that $[W] \cdot \delta_i = w_i$.

$U = U_0$ is the cone in $N_1(Y)_{\mathbb{R}}$ generated by non-negative Minkowski weights. Cones in Δ correspond to torus orbits $\mathcal{O}_\sigma$ which correspond to orbit closures $V(\sigma)$ (which is again a toric variety). Then define U_σ to be the cone in $N_1(Y)_{\mathbb{R}}$ generated by non-negative Minkowski weights on the fan of $V(\sigma)$.

Definition 2. *$tropNE(Y) = cone\ (U_{\sigma \in \Delta} U_\sigma \subseteq N_1(Y)_{\mathbb{R}}$.*

Theorem 1 (Gibney-M)**.**

$$\overline{NE}(Y) \subseteq trop\overline{NE}(Y).$$

The dual formulation is this:

$$\begin{aligned} trop\overline{NE}(Y)^\vee &= i^*([D] \in A_{n-1}(X(\Delta)) \text{iseffectiveforall} i : V(\sigma) \hookrightarrow X(\Delta) && (4)\\ &= i^*\{\cap_{\sigma \in \Delta} pos([D_i] : i \in star(\sigma) \backslash \sigma) + span([D_i] : i \notin star(\sigma)\} && (5)\\ &\subseteq nef(Y) && (6)\end{aligned}$$

. Equality occurs if $i : Y \hookrightarrow X(\Delta)$ is a suitable toric embedding of a Mori dream space.

Response to audience question: Take higher rank toric varieties to get more of the nef cone.

Example (3): for $\overline{M}_{0,n}$, $tropNE(\overline{M}_{0,n}) = \overline{NE}(\overline{M}_{0,n})$ for $n \leq 6$. Don't know for $n \geq 7$.

What does this have to do with tropical geometry?

1.1 Tropical Interpretation

For $X \subseteq (\mathbb{C}^*)^n$ take this and give object $trop(X) \subseteq \mathbb{R}^n$, a d-dimensional (balanced) polyhedral fan with weights on top dimensional cones. Balanced is what we saw above: sum of weighted rays has to be zero.

Why would we want to do this? Tropical variety $trop(X)$ remembers the class $[\overline{X}]$ (the closure of X) in $X(\Sigma)$, intuitively. More precisely, given $C \subset Y \hookrightarrow X(\Delta)$, if $C \cap T \neq \emptyset$ then we can compute $C \cdot \delta_i$ from $trop(C \cap T)$. If $C \cap T = \emptyset$, replace T by some $\mathcal{O}_\sigma$ and repeat. Now if you have a curve in the variety, if you want to know if it's effective, just need to know its tropicalization.

Corollary 1. *Knowing which one-dimensional weighted fans in Δ with $\sum w_i \underline{u}_i = 0$ are $trop(C \cap T)$ and $trop(C \cap \mathcal{O}_\sigma)$ for some $C \subset Y$ is to know $\overline{NE}(Y)$.*

So here's the program:

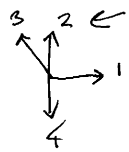
Tropical inverse problem Given a tropical curve $\mathcal{C}$, is $\mathcal{C} = trop(C \cap T)$ for some curve $C \subset Y$? If yes for all $\mathcal{C} \subset \Delta$, then $tropNE(Y) = \overline{NE}(Y)$. (True when $Y = X(\Delta)$ as well, which is a consequence of David Speyer's thesis.) This would give upper bound on cone. However, there are examples (toric embeddings of del Pezzos) with $\mathcal{C} \neq trop(C \cap T)$ for any $C \subset Y$, so this is an honest question.

Could also ask same question for higher-dimensional cycles. Replace "curve" with "cycle" and call it Z. Most hopeful in case of divisors.

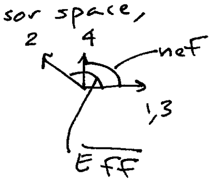
We also get lower bound if Δ is tropicalization of $Y \cap T$... Lower bound and upper bound coincide for polyhedral nef cone.

Please send corrections to taipale at math.umn.edu

1 Fan:

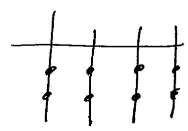


in divisor space,



$N^1(Y)_{\mathbb{R}}$

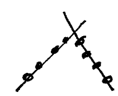
2



$\in \overline{M}_{0,8}$

this is an F-curve

also,



this is a boundary divisor of $\overline{M}_{0,8}$

3/24 D. Maclagan "Tropical bounds on nef cones"
(w/ A. Gibney)

<u>Goal</u>: Use techniques from tropical geometry to bound/compute the nef cone.

Y = proj. var.
($\mathbb{R}$-Cartier)
$N^1(Y) = (\text{divisors})/(\text{numerical equiv.})$, $D \sim D'$ if $(D\cdot C) = (D'\cdot C)$ $\forall$ curves $C \subseteq Y$.

$N^1(Y)_{\mathbb{R}} := N^1(Y) \otimes \mathbb{R}$.

$N_1(Y) = (\text{curves on } Y)/(\text{num. equiv.})$
$N_1(Y)_{\mathbb{R}} = N_1(Y) \otimes \mathbb{R}$.

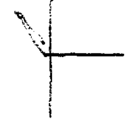
$\overline{\mathrm{Eff}}(Y) \subseteq N^1(Y)_{\mathbb{R}}$ (closure of) cone spanned by <u>effective</u> divisors.

$\overline{NE}(Y) \subset N_1(Y)_{\mathbb{R}}$ (closure of) cone spanned by effective curves.

$$\mathrm{nef}(Y) = \{[D] \in N^1(Y)_{\mathbb{R}} \mid (D\cdot C) \geq 0 \ \ \forall [C] \in \overline{NE}(Y)\}$$
$$= \overline{NE}(Y)^{\vee}$$
(<u>Fact</u>: $\mathrm{nef}(Y) = \overline{\mathrm{amp}(Y)}$.)

<u>Q</u>: Given Y, compute $\overline{\mathrm{Eff}}(Y)$, $\overline{NE}(Y)$, $\mathrm{nef}(Y)$.

(Will give approaches to this problem.)

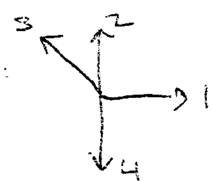
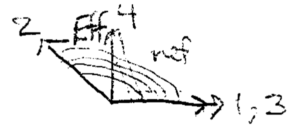
<u>Examples</u>: $Y = X(\Sigma)$ toric var., say simplicial + projective
(e.g. $Y = Bl_p(\mathbb{P}^2)$ )

$N^1(Y)_{\mathbb{R}}$ = spanned by torus-invt divisors.

$\overline{\mathrm{Eff}}(Y)$ = cone gen'd by torus-invt eff. divisors.

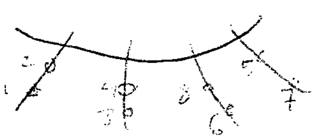
$\overline{NE}(Y)$ = cone gen'd by torus-invt eff. curves.

($\Rightarrow$ polyhedral)

<u>Ex</u>: Σ :  $\Rightarrow$ $N^1(Y)_{\mathbb{R}}$: 

(i.e., $D_1 \sim D_3$, $D_2 \sim D_4 - D_1$)

<u>Ex</u>: $\overline{M}_{0,n}$ = moduli sp of stable genus 0 curves w/ n marked pts. (smooth, dim $= n-3$)



– Has a combinatorial stratification:
(codim k stratum) = (curves with k nodes)

$\left\{ \right.$  $\left. \right\}$ = boundary divisor

<u>General</u>: {curves w/ 1 node} $\longleftrightarrow$ bdy divisors
{curves w/ $n-4$ nodes} $\longleftrightarrow$ curves ("F-curves")

$N^1(\overline{M}_{0,n})_{\mathbb{R}}$ = spanned by bdy divisors.

$\overline{\mathrm{Eff}}(\overline{M}_{0,n}) \supsetneq$ cone gen'd by bdy divisors. (Keel-Vermeire)
$\uparrow n \geq 6$

$N_1(\overline{M}_{0,n})_{\mathbb{R}}$ = spanned by F-curves.

<u>Conj</u> ("F-conjecture" Faber, Fulton):

$\overline{NE}(\overline{M}_{0,n})$ is the cone gen'd by F-curves.

True for $n \leq 7$ by Keel-McKernan.

<u>Note</u>: To know class in $N_1(Y)_{\mathbb{R}}$ of $C \subseteq Y$, suffices to know $(C \cdot \delta_i)$ for $\delta_1, \dots, \delta_s$ spanning $N^1(Y)_{\mathbb{R}}$.

//

Y = proj. var. Fix $i: Y \hookrightarrow X(\Delta)$ = toric var. (assume n.s., but not strictly necessary)

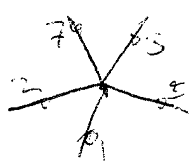
Warning: $X(\Delta)$ need not be complete.

<u>Assumption</u>: $i^*: A_{n-1}(X(\Delta))_{\mathbb{R}} \longrightarrow N^1(Y)_{\mathbb{R}}$ surjective.

Set $\delta_i := i^*(D_i)$, get spanning set.

<u>Lemma</u>: If: $C \subset Y$, $C \cap T \neq \phi$, $C \cdot \delta_i = w_i \geq 0$, then $\sum w_i \underline{u}_i = 0$,

where $\underline{u}_i$ is first lattice pt on i^{th} ray of Δ.



"Fulton/Sturmfels" Minkowski weights.
($\leadsto$ "zero-tension" condition)

<u>Defn</u>: A <u>nonnegative Minkowski weight</u> on Δ is

$$w = (w_i) \in \mathbb{N}^{|\Delta(1)|} \quad \text{s.t.} \quad \sum w_i \underline{u}_i = 0 .$$

$$w \rightsquigarrow [w] \in N_1(Y)_{\mathbb{R}} , \quad [w] \cdot \delta_i = w_i$$

Let:

$\mathcal{U} = \mathcal{U}_0 =$ cone in $N_1(Y)_{\mathbb{R}}$ gen'd by nonneg. Minkowski wts.

<u>Recall</u>: cones in $\Delta \longleftrightarrow$ torus orbits $\mathcal{O}_\sigma$ (inclusion-reversing)
$\longleftrightarrow$ orbit closures $V(\sigma)$

$\mathcal{U}_\sigma$ = cone in $N_1(Y)_{\mathbb{R}}$ gen'd by nonneg. Minkowski wts on $V(\sigma)$

$$\operatorname{trop}\overline{NE}(Y) = \operatorname{cone}\left(\bigcup_{\sigma \in \Delta} \mathcal{U}_\sigma\right) \subseteq N_1(Y)_{\mathbb{R}}$$

<u>Thm</u> (Gibney-M.): $\overline{NE}(Y) \subseteq \operatorname{trop}\overline{NE}(Y)$.

Dually, $\operatorname{trop}\overline{NE}(Y)^\vee = i^*\Big(\{[D] \in A_{n-1}(X(\Delta)) ,$ effective on all $i_\sigma : V(\sigma) \hookrightarrow X(\Delta)\}\Big)$

$$= i^*\Big\{\bigcap_{\sigma \in \Delta} \operatorname{pos}\left([D_i] \mid i \in \operatorname{Star}(\sigma) \setminus \sigma\right) + \operatorname{span}\{[D_i] \mid i \notin \operatorname{Star}\}\Big\}$$

$$\longrightarrow \quad \subseteq \operatorname{nef}(Y)$$

Have <u>equality</u> if $Y \xrightarrow{i} X(\Delta)$ is a suitable toric embedding of a Mori dream space.

<u>Rmk</u>: Want to take emb. in toric var. w/ larger Picard rank...

Ex: $\overline{M}_{0,n}$. $\operatorname{trop}\overline{NE}(\overline{M}_{0,n}) = \overline{NE}(\overline{M}_{0,n})$, $n \le 6$.

$n \ge 7$??

//

Tropical interpretation:

$X^{d} \subseteq (\mathbb{C}^*)^n \rightsquigarrow \operatorname{trop}(X) \subseteq \mathbb{R}^n$, $\operatorname{trop}(X) = \Sigma$,
d-dim'l polyhedral fan, w/ weights on top-dim'l cones.
(balanced)

Roughly: $\operatorname{trop}(X)$ knows $[\overline{X}]$ in $X(\Sigma)$, $\overline{X}$ = closure in $X(\Sigma)$.

Given $C \subset Y \hookrightarrow X(\Delta)$,

- if $C \cap T \neq \emptyset$; can compute $C \cdot \delta_i$ from $\operatorname{trop}(C \cap T)$
- if $C \cap T = \emptyset$, replace T by some O_σ.

$*$ = "tropical curve"

Cor: Knowing which one-dim'l weighted fans$^* \subset \Delta$ with $\sum w_i v_i = 0$
are $\operatorname{trop}(C \cap T)$ for some $C \subset Y$
(or $\operatorname{trop}(C \cap O_\sigma)$)

$\equiv$ knowing $\overline{NE}(Y)$.

//

Tropical Inverse Problem

Given tropical curve $\mathcal{C}$, is $\mathcal{C} = \operatorname{trop}(C \cap T)$
for some curve $C \subset Y$?

If yes, (for all $\mathcal{C} \subset \Delta$), then $\operatorname{trop}NE(Y) = \overline{NE}(Y)$.

- True for $Y = X(\Delta)$.
- There are examples (toric embeddings of del Pezzo surfaces), with $\mathcal{C} \neq \operatorname{trop}(C \cap T)$ for any $C \subset Y$.
- Can ask same for higher-dim'l cycles.