Notetaker Checklist Form - *MSRI*

Name : David Andersen

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I hearby permit MSRI to release my name as the Notetaker. (X) YES ( ) NO

Talk Title/Workshop assigned to:
Boundary complexes of varieties
Combinatorial, Enumerative, and Toric Geometry

Lecturer (Full name): Sam Payne

Date & Time of Event: 3/25/09, 9:30 am

Check list:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the notetaker, and that you will need to make copies of their own notes, if any.

( ) Obtain ALL presentation materials from lecturer. This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this. Either e-mail this to notes@msri.org or obtain a USB stick from the computing department room 214.

( ) Take down ALL notes from media provided (blackboard, overhead, etc.)

( ) Gather ALL other lecture materials, i.e . handouts.

( ) Scan ALL materials in PDF scanner and email to notes@msri.org, in **Subject** please write in Lecturer's name and Talk title .

Please have either the lecturer/yourself, fill in the following when lecture is done:

1. List 6-12 lecture keywords: weight filtration, homology of varieties, boundary complex, log resolution

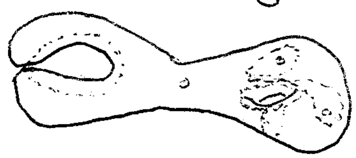
2. Please summerize the lecture in 5 or fewer sentences.
The boundary complex is a Δ-complex associated to an algebraic variety by choosing a weak log resolution and compactification. Its homotopy type is independent of choices, so it can be used to combinatorially construct (old and new) invariants of X.

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Director of IT (rm 214)

3/25 S. Payne, "Boundary complexes of varieties"

Ex:

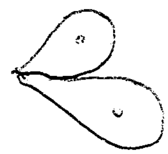
C = punctured singular curve / $\mathbb{C}$:



Deligne: Filtrations on $H^1(C; \mathbb{Q})$

$$W_0 H^1(C) \subseteq W_1 H^1(C) \subseteq W_2 H^1(C) = H^1.$$

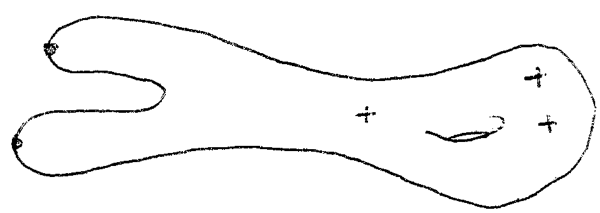
1-cycles:

 ,  ,  } basis for H_1

gen. for W_0 gen for W_1/W_0 gen for W_2/W_1

$\Rightarrow$ dual basis for H^1. ($h^1(C) = 5$)

Let C' = (smth) compactification of normalization:

C': 

$$W_0 H^1(C) \cong \tilde{H}^0(\,\dot{\cdot}\,) \qquad W_2/W_1 \cong \tilde{H}_0(+\,\dot{+}\,)$$

$$W_1/W_0 \cong H^1(C')$$

X = alg var / $\mathbb{C}$, $\dim X = n$.

$\pi : \tilde{X} \longrightarrow X$ <u>weak log resol.</u> : $\tilde{X}$ = smth, $E = \mathrm{Exc}(\pi)$ SNC divisor.

X' = log compactification of $\tilde{X}$ w.r.t. E :

$$\left.\begin{array}{l} X' \setminus \tilde{X} =: \partial X', \\ \partial X' \cup E \end{array}\right\} = \text{SNC divisors}$$

→ Build complex :

<u>Dual complexes</u>

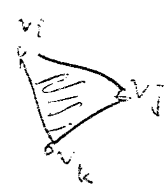
D = SNC divisor $D_1, \dots, D_r$ = components

$\Delta(D)$ has

⇒ <u>vertices</u> $v_1, \dots, v_r$

<u>edges</u> $\underset{v_i}{\circ}\overset{e_Y}{\text{———}}\underset{v_j}{\circ}$, for $Y \subseteq D_i \cap D_j$ irred. comp.

(Note: can have >1 "cell" w/ same vertices ⇒ Δ-cx, not simplicial)

<u>faces</u> f_Z  $Z \subseteq D_i \cap D_j \cap D_k$ irred comp.

<u>Boundary cx</u> : $\Delta(\partial X')$

<u>Thm</u> (Hacking 2008) (–) :

There is a natural isom. $\tilde{H}_{k-1}(\Delta(\partial X'), \mathbb{Q}) \cong gr_{2n} H^{2n-k}(X, \mathbb{Q})$

$$gr_{2n} = W_{2n}/W_{2n-1}$$

<u>Cor</u> : Rational homology of $\Delta(\partial X')$ is indep't of all choices.

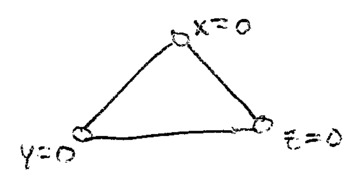
Rmk: blowing up strata is related to barycentric subdiv's of strata... more later.

Thm(-): The homotopy type of the bdy cx $\Delta(\partial X')$ is independent of all choices.

Rmk: Some of these invariants of X were known; some not studied, e.g., integral homotopy gp.

Examples: $X = \mathbb{C}^* \times \mathbb{C}^*$. Compactify: $\mathbb{P}^2 \supseteq X$
$\Rightarrow \mathbb{P}^2 \setminus X$ = union of 3 coord lines.

$\Delta(\mathbb{P}^2 \setminus X) =$



Other compactifications:

$\Delta((Bl_{[0,0,1]}\mathbb{P}^2) \setminus X)$:

$\Delta((Bl_{[1,1,0]}\mathbb{P}^2) \setminus X)$:

Stepanov's Lemma (2006):
$D \subseteq X$ SNC divisor $\quad Z \subseteq X$ smth subvar w/ SNC w.r.t. D
(meets D "as transversally as possible")

π = Blowup of X along Z.

Then there is a homotopy equivalence
$\Delta(\pi^{-1}(D)) \simeq \Delta(D)$.

(3 cases: $Z \not\subset$ any stratum (no change); Z = stratum (barycentric subdiv); $Z \subsetneq$ stratum (glue on cone))

<u>Sketch pf of homotopy invariance</u> of $\Delta(\partial X')$:

$$\pi_1 : \tilde{X}_1 \longrightarrow X \qquad\qquad \pi_2 : \tilde{X}_2 \longrightarrow X$$

$$E_1 \qquad\qquad E_2$$

$$\begin{array}{ccc} X_1' & & X_2' \\ \uparrow & & \uparrow \\ Y_1 & \dashrightarrow & Y_2 \end{array}$$

principalize $\overline{\pi_2(E_2)} \subset X_1$

principalize $\overline{\pi_1(E_1)} \subset X_2'$

(complements of exc. div. + bdy are same)

apply toroidal weak factorization (AKMW 2002)

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<u>Weight filtrations</u> (+ combinatorial geometry)

(all cohom. w/ $\mathbb{Q}$-coeffs)

$$W_0 H^k(X) \subseteq W_1 H^k(X) \subseteq \cdots \subseteq W_{2k} H^k(X) = H^k(X;\mathbb{Q}).$$

If X is smooth + complete, then H^k has "pure weight" k:

$$0 = W_{k-1} H^k \subseteq W_k H^k = H^k.$$

If D = SNC divisor in X = smth complete, can build combinatorial cx of $\mathbb{Q}$-vector spaces

$$0 \to \bigoplus_i H^j(D_i) \xrightarrow{d_0} \bigoplus_{i_0 < i_1} H^j(D_{i_0} \cap D_{i_1}) \xrightarrow{d_1} \bigoplus_{i_0 < i_1 < i_2} \cdots \longrightarrow \cdots$$

j^{th} graded piece $gr_j H^{j+k}(D) = W_j / W_{j-1} = \ker d_k / \operatorname{im} d_{k-1}$

Taking $j = 0$, this cx computes $H^*(\Delta(D))$.

(<u>Rmk</u>: Probably can assume X = rat'l'y smth for these statements)

Weight restrictions :

(1) $W_{2n} H^k(X) = H^k(X; \mathbb{Q})$.

(2) $W_{k-1} H^k(X) = 0$ if X = smth.

(3) $W_k H^k(X) = H^k(X; \mathbb{Q})$ if X = complete.

$$W_k H^k(X) = \ker\left(H^k(X) \to IH^k(X) \right) \qquad \text{(A. Weber, 2004)}$$

Long exact sequences : (of pairs; mapping cylinder for proper morphisms)

Pairs : X' = smth complete, $\partial X'$ = SNC divisor

$$\to H^{k-1}(X') \to H^{k-1}(\partial X') \to H^k_c(X) \to H^k(X') \to \cdots$$

$X = X' \setminus \partial X'$

and this descends to exact seq

$$\to gr_j H^{k-1}(X') \to gr_j H^{k-1}(\partial X') \to gr_j H^k_c(X) \to gr_j H^k(X') \to \cdots$$

"Poincaré duality" : If X = smth,

$$gr_j H^k_c(X) \cong \left(gr_{2n-j} H^{2n-k}(X) \right)^\vee$$

Sketch pf of $\tilde{H}_{k-1}(\Delta(\partial X')) \cong gr_{2n} H^{2n-k}(X)$.

$\pi : \tilde{X} \to X$ weak log resol., $E = \mathrm{exc}(\pi)$, $V = \pi(E)$.

Mayer-Vietoris :

$$\to H^{k-1}(E) \to H^k(E) \to H^k(\tilde{X}) \oplus H^k(V) \to H^k(E) \to \cdots$$

$$\Longrightarrow gr_{2n} H^{2n-k}(X) \cong gr_{2n} H^{2n-k}(\tilde{X}) \qquad (E \text{ has dim } n-1)$$

$\Rightarrow$ Assume X is smooth, X' = log compactification.

PD: $gr_{2n} H^{2n-k}(X;\mathbb{Q}) \cong W_0 H^k_c(X)^\vee$

Using $\to H^{k-1}(X') \to H^{k-1}(\partial X') \to H^k_c(X) \to H^k(X') \to \cdots$
and vanishing for W_0 for sm. vars, get

$$\cong W_0 \tilde{H}^{k-1}(\partial X')^\vee$$
$$\cong \tilde{H}^{k-1}(\Delta(\partial X'))^\vee$$
$$\cong \tilde{H}_{k-1}(\Delta(\partial X')). \quad \boxtimes$$

Case X = affine:

Thm: (Andreotti-Frankel 1959; Karčjauskas 1977)

X has the homotopy type of an n-dim'l CW complex.

$\Rightarrow \Delta(\partial X')$ has rational homology of a wedge of spheres, of dimension $n-1$.

~~Has homotopy type of that of X = com~~

Has homotopy type of wedge of spheres for X = (complement of hyperplane arr.) (Ardila-Klivans)

Warning: Example (Kollár): $T = (\mathbb{C}^*)^n$, $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

$\Delta(\partial X') \cong S^{n-1}$

$T/\sim$, for $t \sim t^{-1}$; $\partial(X'/\sim) \cong \mathbb{RP}^{n-1}$.