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Title: Compactifications of Subvarieties of Tori

Keywords: toric variety, tropical variety, log minimal variety, moduli of del Pezzo surfaces

Summary: A machinery is developed that relates submanifolds in abelian varieties to algebraic tori. Tropicalizations of schön varieties are related to the fans of toric varieties in this theory, and results on log minimality of schön varieties are presented. Moduli of del Pezzo surfaces are treated in an example at the end.

1.1 Submanifolds in abelian varieties

Theorem 1. *$Y \subset X$, Y smooth and X abelian. Then*

- *if Y has general type, then K_Y is ample*
- *if Y is not of general type, then there exists an abelian subvariety $H \hookrightarrow X$ that preserves Y and $Y/H \hookrightarrow X/H$ and $K_{Y/H}$ is ample.*

This was conjectured by Ueno in 1975 and it was proved by Ziv Ran in 1984. A lot of proofs are known, by Zak, Debarre, and others, and this may be an oversimplification as it's only the tip of the iceberg when it comes to results on the Gauss map. But we'll relate this to algebraic tori.

Sketch due to Mori. Notice K_y is globally generated. Suppose K_Y is big (for Y of general type). Then can write $K_y = A+E$ for A ample and E effective. Argue by contradiction: suppose K_Y is not ample. Then there exists a curve $C \subset Y$ such that $K_Y \cdot C = 0$. If you look at $(K_Y + \epsilon E) \cdot C = \epsilon E \cdot C = -\epsilon A \cdot C < 0$, but $\epsilon << 1$ implies $(Y, \epsilon E)$ is klt (Kawamata log terminal). That implies by Cone theorem we can assume C rational, but $C \subset X$ which gives a contradiction. ☐

1.2 Schön varieties

Algebraic tori not compact, so need to deal with issue of smoothness at infinity. Let $Y \subset T$ where T an algebraic torus and Y a closed subvariety. We'll consider various toric varieties X of T. $\overline{Y}$ will always be the closure of Y in X.

Consider the multiplication map $\psi : \overline{Y} \times T \to X$ (or the structure map from $\overline{Y} \to X/T$ the quotient stack*).

Definition 1. *Y is called sch´on if there exists a toric variety X such that $\overline{Y}$ is proper and ψ is smooth and surjective.*

(We don't assume X proper.) Surjectivity simply means that $\overline{Y}$ intersects all torus orbits, and smoothness gives transversality. For example, if X is smooth and Y is schön, then $\overline{Y}$ is also smooth and the induced boundary $B = \overline{Y} \backslash Y$ has simple normal crossings.

Definition 2. *The tropicalization of Y is $trop(Y) = \{w \in N \otimes Q | in_w I$ contains no monomials$\}$. N is lattice of one-parameter subgroups of T, $I \hookrightarrow k[T]$ is the vanishing ideal of Y.*

Theorem 2. *If $Y \hookrightarrow T$ is schön, that is, ψ is smooth, for some toric variety with fan F, then F is supported on $trop(Y)$.*

There is a striking new result by (Luxton, Qu 09): in this case ψ is smooth for any fan supported on $trop(Y)$.

If the structure map is smooth for a toric variety then it will be for any refinement, but these guys proved it for coarsenings as well!

Definition 3. *A smooth variety Y is called log minimal if for one (and hence any) compactification $\overline{Y}$ with SNC boundary B, the linear system $|n(K_{\overline{Y}} + B)|$ gives an embedding of Y for all n sufficiently large and divisible.*

In this case Y has a conjectural compactification by log canonical model $\overline{Y}_{lc} = Proj \oplus_{n \geq 0} H^0(\overline{Y}, n(K_{\overline{Y}} + B))$ (BCHM and abundance).

Theorem 3 (Hacking, Keel, T)**.** *Let $Y \hookrightarrow T$ be a schön subvariety. Then there are two cases:*

- *If Y is of log general type, then Y is log minimal*
- *If not, then there exists a subtorus $H \subset T$ that preserves Y and Y/H is log minimal.*

Sketch. Let X be a smooth toric variety of T. Then we have compactification $(\overline{Y}, B)$ which is smooth and has normal crossings. $(K_{\overline{Y}} + B)$ is globally generated. Suppose that Y is not log minimal. Then there exists a curve $C \subset \overline{Y}$ such that $C \cap Y \neq \emptyset$ and $(K_{\overline{Y}} + B) \cdot C = \emptyset$. Let $\ell = -K_{\overline{Y}} \cdot C = B \cdot C > 1$, because there is no morphism $\mathbb{A}^1 \to T$. By bend and break, can again assume C is rational. (So here picture: can translate C so that it passes through identity element $e \in T$, and C also lies in $\overline{Y}$.) There exists an $(\ell - 1)$-dimensional locus $\overline{S} \subset \overline{Y}$ spanned by deformations of C through e.

Claim: $S = \overline{S} \cap T$ is an algebraic subtorus. Why?

$C \hookrightarrow \overline{Y}$, and for $C_0 = C \cap T$, $C_0 \hookrightarrow T$. $C_0 = \mathbb{P}^1 \setminus \{p_1, \cdots, p_\ell\}$. This inclusion factors through $\mathbb{G}_m^{-1}$. This works for all curves in the family, and algebraic subtori do not deform, so $S = \mathbb{G}_m^{-1}$. Last page of arguments shows that translations by S preserve Y (by deformation theory). $\square$

The theorem also applies to strata of $\overline{Y}$, if $K_{\overline{Y}} + B$ is not ample. We can hope that $\overline{Y}_{\ell c}$ lies in the toric variety.

Definition 4. *Y is called hübsch if Y is schön and $\overline{Y}_{\ell c}$ is a subvariety of a toric variety.*

Corollary 1. *If Y is hübsch, then $trop(Y)$ admits the coarsest fan structure (any other is a refinement), which gives $\overline{Y}_{\ell c}$, such that the following is true: for all $\sigma \hookrightarrow trop(Y)$, $star(\sigma)$ is not preserved by a translation by a linear subspace. If Y is schön and this combinatorial condition is satisfied, then Y is hübsch if positive-dimensional strata of $\overline{Y}$ are connected.*

Under this condition, Hacking proved that if you take the link of the origin of this tropicalization — $link_0\ trop(Y)$ — it has only nontrivial reduced homology in top degree.

1.3 Complicated example.

Brief sketch of contents of 50-page paper: Moduli of del Pezzo surfaces by Hacking-Keel-T. Y^n is the moduli space of marked del Pezzo surface of degree $9 - n$. A marking of S is $K_S^\perp \hookrightarrow Pic(S) \cong Q(E_n)$ root lattice of E_n. $w(E_n)$ acts on Y^n by permuting markings.

$\mathcal{A}_n$ is the $w(E_n)$ lattice freely generated by symbols $[\pm\alpha]$ for pairs of opposite roots.

Definition 5. $M_n = \{\sum n_\alpha[\alpha] | \sum n_\alpha \alpha^2 = 0 \textit{ in } Sym^2 Q(E_n)\}$

Lemma 1. *Y^n is a closed subvariety of an algebraic torus with character group M_n. M_n is a an irreducible $w(E_n)$ module of dimension 0, 5, 15, 35, 84.*

Take the dual map $\mathcal{A}_n \to N_n$, where N_n is where the tropicalization is supposed to live. Then for all root subsystem $\theta \subset E_n$, let $\psi(\theta)$ be a primitive generator of the ray spanned by $\sum_{\alpha\in\theta^+} \psi(\alpha)$.

Theorem 4. *$Y^6 \hookrightarrow \mathbb{G}_m^{15}$ is hübsch, $trop(Y)$ is spanned by rays $\psi(A)$ and $\psi(A_2 \times A_2 \times A_2)$ for all subsystems of this type $\overline{Y}^6_{\ell c}$ is smooth, has normal crossings, is isomorphic to Naruki cross-ratio variety. (Cones – collection of pairwise orthogonal or nested subsystems.)*

$Y^7 \hookrightarrow \mathbb{G}_m^{35}$ is hübsch, $trop(Y)$ is spanned by $\psi(A_1)$, $\psi(A_2)$, $\psi(A_7)$, $\psi(A_3 \times A_3)$ with the same description of cones with one exception: if char $k \neq 2$, one has to exclude cones $A_1 \times \cdots A_1 \hookrightarrow E_7$ where multiplication is k-fold. If char $k = 2$, one has to barycentrically subdivide it. $\overline{Y}^7_{\ell c}$ is smooth, has normal crossings, and isomorphic to Sekiguchi cross-ration.

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① Submanifolds in abelian varieties

Thm: $Y_{sm} \hookrightarrow X_{abelian}$

$\Rightarrow$ (1) if Y has general type, K_Y is ample.

(2) if Y is not of gen'l type, there is an abelian subvar. $H \subset X$, preserving Y, and $Y/H \hookrightarrow X/H$, with $K_{Y/H}$ ample.

(Conjectured by Ueno ~1975; pf by ZivRan ~1984, other pfs by Zak, Debarre, …
$\to$ related to more gen'l results on Gauss map.)

Sketch (Mori): K_Y is globally generated. ($\leadsto$ Gauss map $= \varphi_{|K_Y|}$)
Suppose K_Y is big. so $K_Y = \underset{\text{ample}}{A} + \underset{\text{effective}}{E}$

For contradiction, assume $K_Y \neq$ ample. Then $\exists$ curve $C \subseteq Y$ s.t. $K_Y \cdot C = 0$

$\Rightarrow (K_Y + \varepsilon E) \cdot C = \varepsilon E \cdot C = -\varepsilon A \cdot C < 0.$

But $\varepsilon \ll 1 \Rightarrow (Y, \varepsilon E)$ is klt.

So by Cone Thm, can assume C is rational.

But then $C \subset X$ rat'l curve in abelian variety $\times\!\!\!\times$.

② Schön varieties

$\underset{\text{closed subvar.}}{Y} \subset \underset{\text{alg. torus}}{T}$ $\quad$ X = toric var. for T

$\overline{Y}$ = closure of Y in X.

Consider multiplication map $\psi : \overline{Y} \times T \longrightarrow X$

(or structure map $\overline{Y} \xrightarrow{\psi} X/T$)

<u>Defn</u>: Y is called schön if $\exists$ toric variety X s.t. $\overline{Y}$ is proper, and ψ is smooth + surjective.

Surjectivity $\longleftrightarrow$ $\overline{Y}$ intersects all torus orbits in X.

Smoothness $\longleftrightarrow$ transversality.

<u>Ex</u>: If X is smth, Y schön, then $\overline{Y}$ = smooth, $B = \overline{Y} \setminus Y$ SNC.

(<u>Rmk</u>: for $Y \subset T$ hypersf, schön = nondeg. wrt. Newton polyhedron.)

<u>Defn</u>: $\operatorname{trop}(Y) := \{ w \in N \otimes \mathbb{R} \mid in_w(I)$ contains no monomials $\}$

(N = lattice $\operatorname{Hom}(\mathbb{C}^*, T)$, $I \subset k[T]$ ideal of Y)

<u>Thm</u>: If $Y \hookrightarrow T$ is schön, i.e., ψ is smth for some toric variety w/ fan $\mathcal{F}$, then $\mathcal{F}$ is supported on $\operatorname{trop}(Y)$.

Moreover (Luxton–Qu 2009), in this case, ψ is smth for <u>any</u> fan supported on $\operatorname{trop}(Y)$.

<u>Def^n</u>: A smth variety Y is <u>log minimal</u> if for some ($\Leftrightarrow$ any) smth compactification $\overline{Y}$ with s.n.c. bdy B, $|n(K_{\overline{Y}}+B)|$ gives an embedding of Y $\forall$ n suff. divisible.

~~In this case~~,

In this case, Y has a conjectural compactification, log canonical model

$$\overline{Y}_{lc} = \mathrm{Proj} \bigoplus_{n\geq 0} H^0(\overline{Y}, n(K_{\overline{Y}}+B)).$$

([BCHM] + abundance)

<u>Thm</u> (Hacking, Keel, T.): Let $Y \hookrightarrow T$ be schön. Then

(1) If Y is of log general type, then Y is log minimal

(2) If not, then $\exists$ $H \hookrightarrow T$ subtorus preserving Y, s.t. Y/H is log minimal.

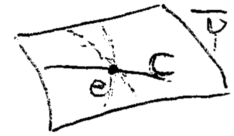
<u>Sketch</u>: Let X be a smth toric variety, for T. $(\overline{Y}, B = \overline{Y} \setminus Y)$ compactification.

~~Ex~~ $K_{\overline{Y}}+B$ is globally generated.

Suppose Y not log minimal. Then $\exists$ curve $C \subsetneq \overline{Y}$ s.t. $C \cap Y \neq \emptyset$ and $(K_{\overline{Y}}+B)\cdot C = 0$.

so $\ell := -K_{\overline{Y}} \cdot C = B \cdot C > 1$ (b/c there's no $\mathbb{A}^1 \to T$)

By bend+break, can assume C = rat'l.



, $e \in T$ identity

There's $(\ell-1)$-dim'l locus $\overline{S} \subset \overline{Y}$ spanned by deformations of C through e,

<u>Claim</u>: $S = \overline{S} \cap T \hookrightarrow T$ is an alg. subtorus.

Why?

$$\begin{tikzcd} C \arrow[r, hookrightarrow] & \overline{Y} & \rightsquigarrow & C_0 \arrow[r, hookrightarrow] \arrow[d] & T \\ & & \mathbb{P}^1 \setminus \{p_1, \dots, p_\ell\} \arrow[u, phantom, "="] & \mathbb{G}_m^{\ell-1} \arrow[ur] & \end{tikzcd} \qquad (C_0 = C \cap T)$$

This works for all curves in the family, and algebraic subtori don't deform.

$$\Rightarrow \quad S = \mathbb{G}_m^{\ell-1}.$$

Then show S preserves Y (by deformation theory.)

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Rmk: can always get $\overline{Y}_{lc}$ in a <u>non</u>-separated toric var.

The thm also applies to strata of $\overline{Y}$.

If $K_{\overline{Y}} + B \neq$ ample, can hope that $\overline{Y}_{lc}$ lives in the toric variety.

<u>Defn</u>: Y is called <u>hübsch</u> if Y is schön and $\overline{Y}_{lc}$ is a subvariety of a toric variety.

<u>Cor</u>: If Y is hübsch, then $\mathrm{trop}(Y)$ admits the coarsest fan structure (which gives $\overline{Y}_{lc}$) such that:

(*) for any cone $\sigma \subset \mathrm{trop}(Y)$, $\mathrm{Star}(\sigma)$ is not preserved by translation by any linear subspace.

If Y is schön and (*) is satisfied, then Y is hübsch if positive-dimensional strata of $\overline{Y}$ are connected.

(idea: different conn. comps might be preserved by different linear translations)

Under this condition, Hacking proved:

$\mathrm{link}_0(\mathrm{trop}(Y))$ has only non-trivial reduced homology in top degrees.

Example (moduli of del Pezzo surfaces) (Hacking-Keel-T.)

Y^n = moduli of marked del Pezzo's of degree $9-n$.

marking: $(K_S^\perp \hookrightarrow \mathrm{Pic}\,S) \cong$ root lattice of E_n.

$W(E_n)$ acts on Y^n by permuting markings.

Let A_n be the $W(E_n)$-lattice freely generated by symbols $[\pm\alpha]$ for pairs of opposite roots.

Def: $M_n = \{\sum n_\alpha[\alpha] \mid \sum n_\alpha \alpha^2 = 0 \text{ in } \mathrm{Sym}^2(\overbrace{Q(E_n)}^{\text{root lattice}})\}$

Lemma: Y^n is a closed subvariety of an alg. torus with character gp M_n.

M_n is an irred. $W(E_n)$-module of rank 0, 5, 15, 35, 84.

Take dual map $A_n \longrightarrow N_n$ $\rightsquigarrow$ receptacle for $\mathrm{trop}(Y)$.

For any root subsystem $\Theta \subset E_n$, let $\psi(\Theta)$ be a primitive generator of the ray spanned by $\sum_{\alpha\in\Theta^+}\alpha$.

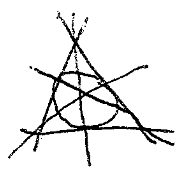
Thm: $Y^6 \hookrightarrow \mathbb{G}_m^{15}$ is hübsch.

- trop(Y) is spanned by rays $\Psi(A_1)$ and $\Psi(A_2 \times A_2 \times A_2)$ for all subsystems of this type.
- cones $\Leftrightarrow$ collections of pairwise orthogonal or nested subsystems.
- $\overline{Y}^6_{lc}$ is smth, has normal crossings bdy, is isom. to Naruki cross-ratio variety.

- $Y^7 \hookrightarrow \mathbb{G}_m^{35}$ is hübsch;
- trop(Y) spanned by $\Psi(A_1)$, $\Psi(A_2)$, $\Psi(A_7)$, $\Psi(A_3 \times A_3)$, with same description of cones,

→ with one exception:

- if char$(k) \neq 2$, include cones for $\overbrace{A_1 \times \cdots \times A_1}^{7} \hookrightarrow E_7$.
- if char$(k) = 2$, barycentrically subdivide.

( blowup pts, contract lines $\Rightarrow$ del Pezzo with 7 A_1 singularities)

- $\overline{Y}^7_{lc}$ is smth, with n.c. bdy, isom. to Sekiguchi cross-ratio.