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Title: Quantum K theory of Grassmannians

Keywords: Grassmannian, quantum K-theory, Gromov-Witten invariant, Gromov-Witten variety

Summary: The quantum K-theory of Grassmannians is introduced and discussed. "Quantum = classical" is a main result: Buch-Kresch-Tamvakis proved that quantum cohomology of Grassmannians is equivalent to interesection theory on the two-step flag variety, and the author and Buch have extended this to quantum K-theory. Rationality of Gromov-Witten varieties is touched on.

Quantum K-theory of Grassmannians! Joint work with Anders Buch, and the paper is available on the arXiv.

What is this object called quantum K-theory? For definition part, we'll look at $X = G(p,m) = \{V \subset \mathbb{C}^m \,|\, \dim V = p\}$. There is a Borel subgroup $B \subset GL(m)$ which acts on X (B is upper triangular matrices) and B-orbits give a stratification $X = \coprod X^0_\lambda$, where X^0_λ are Schubert cells and $\lambda = (\lambda_1 \geq \cdots \lambda_p \geq 0)$ which gives a Young tableau in an $(p \times (m-p))$ rectangle.

Closures $\overline{X}^0_\lambda$ are Schubert varieties, and $\sigma_\lambda = [X_\lambda] \in H^{2|\lambda|}(X, \mathbb{Z})$, where $|\lambda| = \lambda_1 + \cdots + \lambda_p$.

Facts:

- $H^*(X)$ has a $\mathbb{Z}$-basis $\{\sigma_\lambda\}$
- $\sigma_\lambda \cdot \sigma_\mu = \sum c^\nu_{\lambda\mu}$, $|\nu| = |\lambda| + |\mu|$.
- $c^\nu_{\lambda\mu}$ are the LR coefficients, defined by $\#\{g_1 X_\lambda \cap g_2 X_\mu \cap g_3 (X_\nu)^\vee\} \geq 0$

1.1 Quantum Cohomolgoy

- $QH^*(X)$ is a graded $\mathbb{Z}[q]$ algebra, deg $q = m$, with $\mathbb{Z}[q]$-basis $\{\sigma_\lambda\}_{\lambda \subset rectangles}$
- $\sigma_\lambda * \sigma_\mu = \sum_{\nu,d} c^{\nu,d}_{\lambda,\mu} q^d \sigma_\nu$ (note Littlewood-Richardson coefficients)
- $c^{\nu,d}_{\lambda,\mu} = \#\{f : (\mathbb{P}^1, 0, 1, \infty) \to X, \deg f = d | f(0) \in g_1 X_\lambda, f(1) \in g_2 X_\mu, f(\infty) \in g_3 (X_\nu)^\vee\}$ and these are the three-point Gromov-Witten invariants.
- need to satisfy dimension contraints because else I may get infinitely many maps (a positive-dimensional space of maps) and then by definition the GWI is equal to zero.

Note: grading implies $|\lambda| + |\mu| = |\nu| + (\deg q) \cdot d \implies$ expected dimension of variety of maps is zero. Many ways to compute these Gromov-Witten invariants, also: algorithms by Bertram, Bertram-Fulton-Ciocan-Fontanine, Vakil, Tamvakis, Coskun...

We'd like to relax these constraints, and we define

Definition 1. *The Gromov-Witten variety is the compactification of*

$$\{f : (\mathbb{P}^1, 0, 1, \infty) \to X,\ \deg f = d,\ f(0) \in g_1X_\lambda, f(1) \in g_2X_\mu, f(\infty) \in g_3X_\nu\}$$

in the Kontsevich moduli space of stable maps $\overline{M}_{0,3}(X, d)$*, and we denote it by* $GW_d(X_\lambda, X_\mu, X_\nu)$*.*

Examples: (1) Say $X = \mathbb{P}^1$, $d = 1$. Map $\mathbb{P}^1$ with $0, 1, \infty$ marked to $\mathbb{P}^1$ with x_0, x_1, x_2 marked: $GW_1(\mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1) = \overline{M}_{0,3}(\mathbb{P}^1, 1$ which has dimension 3. $f([x, y]) = \frac{ax+by}{cx+dy})$. Can see from this that the moduli space is rational. Rationality will be important later.

(2) $GW_1(pt, pt, \mathbb{P}^1)$: 0 goes to 0, 1 goes to 1, and ∞ goes anywhere except 0 or 1. Thus $GW_1(pt, pt, \mathbb{P}^1) = \mathbb{P}^1$.

(3) $GW_1(pt, pt, pt) = pt$. This is what was used to define usual quantum cohomology.

Question, then: Can we find a quantum cohomology theory which corresponds to Gromov-Witten varieties of positive dimension? Answer: Yes. It's going to be quantum K-theory. Look at *picture 1* at the end for a diagram of relationships. In this talk we'll look at $QK(X)$ and $QK_T(X)$.

1.2 K-theory

For $\mathcal{F}$ a coherent sheaf on X, $K(X) = \oplus_{\mathcal{F}}[\mathcal{F}]/\sim$ where relations $\sim$ come from short exact sequences: $[\mathcal{F}] = [\mathcal{G}] + [\mathcal{H}]$ if $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$. $\mathcal{F}$ has a resolution by algebraic vector bundles

$$0 \to E_m \to \cdots \to E_1 \to \mathcal{F} \to 0$$

which gives an algebra structure $[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$. There's also an Euler characteristic

$$\chi : K(X) \to \mathbb{Z}$$

taking $[\mathcal{F}] \mapsto \sum(-1)^i H^i(X, \mathcal{F})$, from which you can build a Poincaré pairing

$$\langle \cdot, \cdot \rangle : K(X) \times K(X) \to \mathbb{Z},$$

taking $\langle [\mathcal{F}], [\mathcal{G}] \rangle = \chi([\mathcal{F}] \cdot [\mathcal{G}])$. Note $\mathcal{O}_\lambda \neq [\mathcal{O}_{X_\lambda}]$.

Examples: $X = \mathbb{P}^1$, $d = 1$, and $\chi(GW_1(\mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1), \mathcal{O}) = \chi(\overline{M}_{0,3}(\mathbb{P}^1, 1), \mathcal{O}) = 1$ since $\overline{M}_{0,3}(\mathbb{P}^1, d)$ is rational.

Definition 2 (Givental)**.** *The K-theoretic Gromov-Witten invariant is defined by*

$$\langle \mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu \rangle_d = \chi(\mathcal{O}_{GW_d(X_\lambda, X_\mu, X_\nu)}).$$

Q: If X is not compact, can we define this? A: Yes, however there is a subtlety. Definition uses evaluation map and we're multiplying in K-theory of moduli space. Need K-theoretic version of Kleiman-Bertini to get the equality above, and Susan Sierra proved that version.

More examples: $X = \mathbb{P}^1$. (2) $\langle \mathcal{O}_{pt}, \mathcal{O}_{pt}\, \mathcal{O} \rangle_1 = 1$ (3) $\langle \mathcal{O}_{pt}, \mathcal{O}_{pt}\, \mathcal{O}_{pt} \rangle = 1$ and this is usual GW invariant.

1.3 Quantum K-theory

Quantum K-theory is *expected* to have the following properties:

- $QK(X)$ is a (filtered) $\mathbb{Z}[q]$-algebra with a $\mathbb{Z}[q]$-basis consisting of $\mathcal{O}_\lambda$
- $\mathcal{O}_\lambda \cdot \mathcal{O}_{mu} = \sum N^{\nu,d}_{\lambda,\mu} q^d \, \mathcal{O}_\nu$
- $N^{\nu,d}_{\lambda,\mu} = \langle \mathcal{O}_\lambda, \mathcal{O}_\mu, (\mathcal{O}_\nu)^\vee \rangle_d$

Important example: $X = \mathbb{P}^1$. $\mathcal{O}_{pt} \cdot \mathcal{O} = \mathcal{O}_{pt} + \langle \mathcal{O}_{pt}\, \mathcal{O}, \mathcal{O}^\vee \rangle q\, \mathcal{O} + \dots$. $\mathcal{O}^\vee = \mathcal{O}_{pt}$ so $\langle \mathcal{O}_{pt}, \mathcal{O}, \mathcal{O}^\vee \rangle_1 = \langle \mathcal{O}_{pt}, \mathcal{O}, \mathcal{O}_{pt} \rangle_1 = 1$. This definition cannot be good, as it has no unit. So we need to fix or change something.

In reality, we have the following:

Definition 3 (Givental-Lee)**.** *$QK(X)$ is a $\mathbb{Z}[[q]]$-algebra with a $\mathbb{Z}[[q]]$-basis consisting of $\mathcal{O}_\lambda$, and the structure constants have corrections as follows:*

$$N^{\nu,d}_{\lambda,\mu} = \langle \mathcal{O}_\lambda, \mathcal{O}_\mu, (\mathcal{O}_\nu)^\vee \rangle_d + corr \tag{1}$$

$$= \langle \mathcal{O}_\lambda, \mathcal{O}_\mu, (\mathcal{O}_\nu)^\vee \rangle_d - \sum_{e>0,d} N^{\alpha,d-e}_{\lambda,\mu} \langle \mathcal{O}_{alpha}, (\mathcal{O}_{nu})^\vee \rangle_e \tag{2}$$

$$= \chi_{\overline{M}_{0,3}(X,d)}(ev_1^*(\mathcal{O}_\lambda) \cdot ev_2^*(\mathcal{O}_\mu) \cdot ev_3^*(\mathcal{O}_\nu)) - \chi_D(ev_1^*(\mathcal{O}_\lambda) \cdot ev_2^*(\mathcal{O}_\mu) \cdot ev_3^*(\mathcal{O}_\nu)). \tag{3}$$

(there are corrections).

Reason for corrections: look at D_1, D_2 in $\mathbb{P}^3$. Then $[\mathcal{O}_{D_1 \cup D_2}] = [O_{D_1} + O_{D_2} - O_{D_1 \cap D_2}$, the inclusion exclusion principle.

1.4 Main results

At least in the case of Grassmannians, here is how to compute the quantum theory.

$X = G(p, m)$. Let d be fixed, $a = \max(p - d, 0)$ and $b = \min(p + d, m)$. See *picture 2* at end of file for diagram.

Theorem 1 (Quantum = classical)**.** *We can translate quantum K-invariants of Grassmannians to the K-theory of the two-step flag variety:*

- $\langle \mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu \rangle_d^{G(p,m)} = \langle \mathcal{O}_{Y_\lambda}, \mathcal{O}_{Y_\mu}, \mathcal{O}_{Y_\nu} \rangle_0^{Fl(a,b;m)}$
- $\langle \mathcal{O}_{(1)}, \mathcal{O}_\mu, \mathcal{O}_\nu \rangle_d^{G(p,m)} = \langle \mathcal{O}_{(1-d)}, \mathcal{O}_{\mu(d)}, \mathcal{O}_{\nu(d)} \rangle_0^{G(b,m)}$ *(**)*

Buch-Kresch-Tamvakis proved this for $QH^*(X)$, Buch-M for $QK_T(X)$

These results extend to cominiscule Grassmannians by results of Chaput, Manivel, Perrin if d is small. Why is this result nice? Right-hand side is computable for these.

We also have quantum-K Pieri and Giambelli rules; will skip to get to sketch of proof.

Sketch. For a map $f : \mathbb{P}^1 \to G(p,m)$, $\deg f = d$. Notions of kernel and span due to Buch. $Ker\ f = \cap_{x\in\mathbb{P}^1} f(x)$ and $Span\ f = Span\{f(x)|x \in \mathbb{P}^1\}$.

Proposition 1 (Buch)**.** *dim $Ker\ f \geq p - d$, dim $Span\ f \leq p + d$, and equality occurs for most curves.*

See *picture 3* for big diagram.

ψ in general has positive-dimensional fibers, and so need the fact that $\psi_*[\mathcal{O}_{Bl_d}] = [\mathcal{O}_Z]$. For that need to know about fibers: cohomology of all fibers vanishes for positive i . The fibers $GW_d(pt, pt, pt)$ are irreducible and rational for $d \geq \max(p, m-p)$, but a good question is when the GW varieties are irreducible and rational. □

Q: are there versions of Grothedieck polynomials for K-theory? A: yes, some analog (recursive) for Grassmannians. But we don't know about flag varieties. Also, all structure coefficients satisfy positivity in Ezra's sense: alternating signs.

Please send corrections to taipale at math.umn.edu

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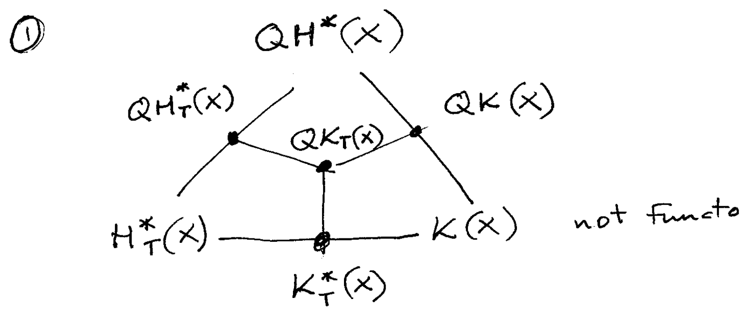
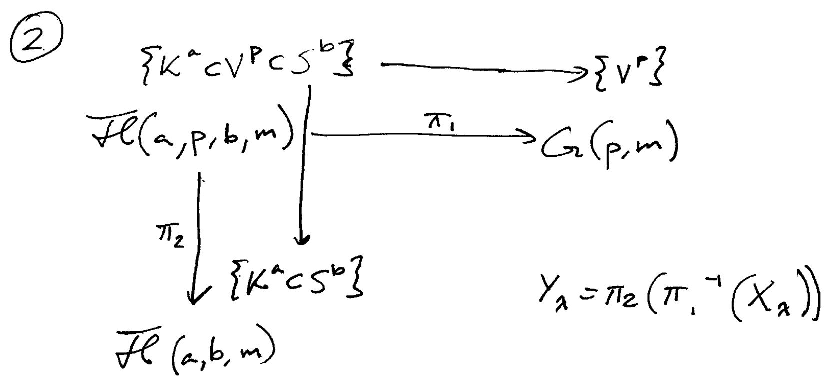
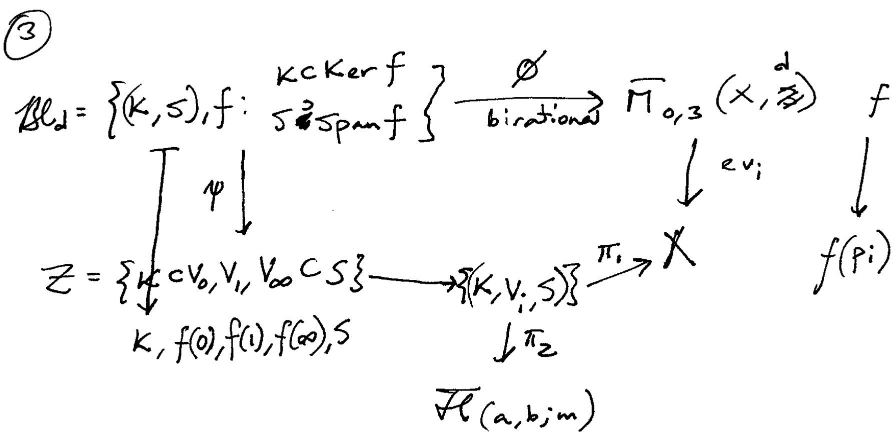


1
QH*(X)
QH*_T(X)
QK(X)
QK_T(X)
H*_T(X)
K(X)
K*_T(X)
not functo



2
{K^a ⊂ V^p ⊂ S^b}
{V^p}
Fl(a,p,b,m)
π_1
Gr(p,m)
π_2
{K^a ⊂ S^b}
Fl(a,b,m)
Y_λ = π_2(π_1^{-1}(X_λ))



3
Bl_d = {(K,S), f : K ⊂ Ker f, S ⊃ Span f}
φ
birational
M̄_{0,3}(X, d)
f
ev_i
ψ
Z = {K ⊂ V_0, V_1, V_∞ ⊂ S}
K, f(0), f(1), f(∞), S
{(K, V_i, S)}
π_1
X
f(p_i)
π_2
Fl(a,b;m)