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Title: K-theoretic Schubert calculus on the affine Grassmannian

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Summary: Three models for the affine Grassmannian are introduced. These models are used to explore the (K-theoretic) Schubert calculus on the affine Grassmannian, with "bonus" and "non-bonus" features of the calculus listed. To end, the speaker's results on K-theory of the affine Grassmannian (with co-authors Schilling, Shimozono) are presented.

Joint with Anne Schilling and Mark Shimozono.

G a simple simply-connected algebraic group over $\mathbb{C}$, B Borel, T torus, $G \supset B \supset T$, and K maximal compact subgroup of G, $T_{\mathbb{R}}$ the real torus.

For an example, think of $SL(n, \mathbb{C})$ containing (upper triangular matrices) containing $(\mathbb{C}^*)^{n-1}$, with compact group $SU(n)$ containing $(S^1)^{n-1}$.

S_n is the Weyl group W, with $S_n \ltimes \mathbb{Z}^{n-1}$ the affine Weyl group $W_{af} = W \ltimes Q^\vee$, and $Q^\vee$ the coroot lattice. (In example, the coroot lattice is n-vectors in $\mathbb{Z}^n$ with sum of entries $= 0$). Note that $T_{af} = T \times \mathbb{C}^*$. A typical root of W_{af} is $\alpha + n\delta$, where n tells you the action of the extra $\mathbb{C}^*$ and α is a usual root.

Gr_G is the affine Grassmannian, and a model for it is $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) = \mathcal{G}/\mathcal{P}$. This quotient is an ind-scheme, a union of finite-dimensional schemes.

(Note: $\mathbb{C}((t)) = \{\sum_{i>-N} a_i t^i | a_i \in \mathbb{C}\} = \cup_{N \leq 0} t^N \, \mathbb{C}[[t]]$.)

Another model for the affine Grassmannian Gr_G is $\Omega K = \{f : S^1 \to K\}$ where f is polynomial, $f(1) = 1$, and this is a topological group.

The last model is the thick affine Grassmannian, an invention of Kashiwara. $X = Gr_G^{thick}$ is a scheme of infinite type. $X = \coprod_{w \in W_{af}/W*} \overset{\circ}{X}_w$ where $\overset{\circ}{X}_w \cong$ Spec$(\mathbb{C}[x_1, \ldots])$.

Look back: $\mathcal{G}/\mathcal{P} = \coprod_{w \in W_{af}/W} \overset{\circ}{X}_w$. $\dim(\overset{\circ}{X}_w) = \ell(w)$, the length of the minimal-length coset representative. Finite-dimensional Schubert cells/varieties $X_w = \overline{\overset{\circ}{X}_w}$.

Schubert varieties and the T-fixed points are labeled by W_{af}/W, which is in bijection with the coroot lattice $Q^\vee$. It is also in bijection with teh cosets W_{af}, and we could label by minimal coset representatives.

How do we get the torus-fixed points? The easiest way is to look in the second model. $Q^\vee$ is the group of homomorphisms $\{\chi : \mathbb{C}^* \to T\} = \chi : S^1 \to T_{\mathbb{R}}$. The T-fixed points correspond to cocharacters.

How do we compare the models? ΩK and $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ are known to be weak homotopically equivalent (Quillen transcribed by Mitchell, Pressley-Segal). Think of loop groups; $G/B = K/T_{\mathbb{R}}$.

There's an interpretation of the affine Grassmannian that mirrors the traditional one, at least for $SL(n)$. Let $G = SL(n, \mathbb{C})$. Then

$$SL(m, \mathbb{C}((t)))/SL(n, \mathbb{C}[[t]]) = \{\text{full rank}\, \mathbb{C}[[t]] - \text{modules}\ \mathcal{L} \in \mathbb{C}((t))^n \text{ of det } 1\} \quad (1)$$

The right-hand side is a lattice. Think of top of LHS as basis for lattice, and bottom of LHS as what I'm allowed to change basis. A typical point in the LHS looks like

$$\begin{pmatrix} t_1^a & 0 & - & 0 \\ 0 & t^{a_2} & - & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & - & t^{a_n} \end{pmatrix}$$

which corresponds to $\mathcal{L} = span_{\mathbb{C}[[t]]}\{t^{a_1}e_1, t^{a_2}e_2, \ldots, t^{a_n}e_n\}$.

Philosophy: When you study Schubert calculus you get a whole package once you decide what space to study. You can look at $H^*, K^*, H^T, K^T, QH, QK, \ldots$. $H^{T_{af}}(\mathcal{G}\,/\,\mathcal{P})$ and $K^{T_{af}}(\mathcal{G}\,/\,\mathcal{P})$ were studied by Kostant and Kumar in the general Kac-Moody setting. They studied topological K-theory, but the claim today is it doesn't matter which version you study (or the model) if we're only interested in the ring.

Claim: H^T, K^T are the same for all three models. (The homotopy equivalence is T-equivariant.)

Why? We can use localization and look at fixed points.

$$H^T(Gr_G) \subset \prod_{Q^\vee} H^T(pt),$$

$$K^T(Gr_G) \subset \prod_{Q^\vee} K^T(pt).$$

(Q: When does this stuff matter, then? A: When you want to know the geometric structure of the basis. Hopf structure is same, which we'll talk about later.)

We're going to use a smaller torus. (This idea is due to Petersen.)

Bonus features of the Grassmannian Gr_G:

- $Gr_G = \Omega K$ is a group. We get extra structure on the rings $H^T(Gr_G)$ and $K^T(Gr_G)$ and can make them Hopf algebras. Bigger torus is not equivariant with respect to the group operation, which is why small torus is important.

- There are natural maps $\mathcal{G}\,/\,\mathcal{B} \to \mathcal{G}\,/\,\mathcal{P}$, but also map going the other way: $\Omega K = \mathcal{G}\,/\,\mathcal{P} \to \mathcal{G}\,/\,\mathcal{B} \cong LK/T$

- $G \hookrightarrow H$ gives $Gr_H \hookrightarrow Gr_H$, which gives $H_*(Gr_G) \to H_*(Gr_H)$.

- For $G = SL(n, \mathbb{C})$, $H_*(Gr_G) = \mathbb{Z}[h_1, \ldots, h_{n-1}] \subset Sym$ (h_i symmetric polynomials). No quotients. ($\Delta h_i = \sum_{j \leq i} h_j \otimes h_{i-j}$)

- (Affine Grassmannian important in geometric representation theory, and has important subvarieties that don't appear in usual case.)

Non-bonus features:

- $\xi^w \xi^u = \sum c^v_{uw} \xi^v \in H^*(Gr_G)$. Can prove structure constants are positive (Kumar-Nori), but would like $c^v_{uw} = \#\{\text{lattices } \mathcal{L} \text{ satisfying certain conditions}\}$.
- Multiplication $\Omega K \times \Omega K \to \Omega K$ is not algebraic.
- The map from the affine grassmannian to affine flag variety, $\Omega K \hookrightarrow LK$, also not algebraic.
- $\mathcal{G} / \mathcal{P}$ is not smooth. What does that mean? Fishel, Goj, Telemann have result that Hodge decomposition fails; can't write this as a union of finite dimensional smooth subvarieties.
- In homology can write Schubert classes as polynomials with no quotient, but the variables have no apparent direct geometric meaning.

1.1 GKM model for $K^T(Gr_G)$

The usual

$$K^T(G/B) = \{f : W \to R(T) = K^T(pt) | f(w) - f(r_\alpha w) \in (1 - e^{-alpha})R(T) \; \forall \; w, \alpha\} \tag{2}$$

$$\subset \prod_w R(T). \tag{3}$$

This doesn't quite work for $K^T(Gr_G)$ because in GKM need one-dimensional edges with different weights, and edges with $r_\alpha, r_{\alpha+\delta}, r_{\alpha+2\delta}$ all have the same weight under small torus action.

Theorem 1. $f : Q^\vee \to R(T)$. *Then*

$$f \in K^T(Gr_G) \iff f((1 - t_{\alpha^\vee})^d t_\lambda) \in (1 - e^\alpha)^d R(T) \; \forall \; d \geq 0, \alpha, \lambda.$$

Proposition 1. $\Delta f(t_\lambda \otimes t_\mu) = f(t_{\lambda+\mu})$

These give a complete description of the Hopf algebra (equivariant K-cohomology).

In last five minutes, give description of dual; $\mathbb{K}$ will be the Kostant-Kumar nil Hecke ring, and it's $\langle T_i, e^\lambda / \sim \rangle$. The relations $\sim$ come from $\mathbb{K}$ acting on $K^T(\mathcal{G} / \mathcal{B})$.

$$K^T(\mathcal{G} / \mathcal{B}) \to K^T(\mathcal{G} / \mathcal{P}_i) \to K^T(\mathcal{G} / \mathcal{B})$$

and the composition of these two arrows is T_i. (Note $e^\lambda \in \mathbb{Z}[P] = R(T)$.

$\mathbb{K}$ also acts dually on K-homology. Petersen's idea is that because of thereverse map get $\Omega K \subset LK$, so ΩK acts on LK/T. Get an action of $K_T(Gr_G)$ on $K_T(\mathcal{G} / \mathcal{B})$.

We'll end with a theorem:

Theorem 2 (LSS, Petersen in homology)**.** *There exists* $k : K_T(Gr_G) \xrightarrow{\sim} Z_{\mathbb{K}}(R(T)) \subset \mathbb{K}$*, where* Z *is the centralizer.*

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