RIMSE in CLAS at ASU:

Research and Innovation in Mathematics and Science Education

Mathematics Education Research: Approaches and Insights for Improving Mathematics Teaching and Learning

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The Improvement of Mathematics Teaching and Learning is a National Priority

- Mathematicians, mathematics educators and policy makers are investing time and resources in various attempts to improve mathematics teaching and learning
- Funding is available nationally and locally to reinvent new approaches to mathematics teaching and learning
- Mathematicians and math departments are embracing the scholarship of mathematics learning and teaching.

Mathematics Education Researchers are Generating Knowledge that Should be Useful to Policy Makers, Curriculum Development and Instructors

However,

This knowledge appears to be having limited impact on mathematics curriculum, test development, and teaching practice. (Why?)

National Documents Have Omitted Qualitative Studies that are Uncovering Important Information About How Students Learn Key Ideas of Algebra

One Example: The Report of the Task Group (of the National Mathematics Advisory Panel) on Conceptual Knowledge and Skills (2008) provides only a listing of topics in algebra that students should know (while providing no basis for this recommendation). Further, it makes unqualified recommendations with no reference to research (mostly about what symbolic manipulations students should be able to do—nothing about what it is important for student to understand). All the while, they make *No* reference to the 20+ years of qualitative work that has uncovered knowledge about how students' conceptions of variable, rate of change, function, equation, etc. emerge.

Questions About Mathematics Learning and Teaching that are Predominant in Debates and National Documents

- Should procedural knowledge proceed conceptual learning?
- Should classrooms be teacher directed or student centered?
- Do students learn better when problems are presented in applied contexts?
- Do students learn more in small classes?
- Does use of technology enhance learning?
  - National Math Panel Final Report: High-quality computer-assisted instruction be implemented with as a tool for developing students automaticity. What did the panel mean by “high quality” software? Under what conditions did the software enhance learning?

Many Central Questions About Student Learning and Teaching Need to Be Addressed in National Debates and Documents

- What does it mean for students to understand “x”?
- What is the process by which students learn “x”?
- What instructional experiences promote the development of effective problem solving behaviors IN STUDENTS?
- What are criteria for a “highly effective” teacher?
- What is the role of the teacher and curriculum in supporting students’ mathematical development?
- What professional supports are needed to produce “highly effective” teachers? What does it mean to be an “highly effective” teacher?
- What attributes of homework are effective for facilitating knowledge construction
- How much and what kinds of “repeated reasoning” experiences are needed for learning ideas of average rate of change, angle measure, limit, FTC?

Some Areas in Which Consensus is Emerging

- Students need to acquire procedural knowledge, conceptual knowledge, and problem solving abilities
  - **Problem:** National documents (including *The Final Report of the National Mathematics Advisory Panel (2008)*, and *Principles and Standards for Secondary Mathematics (2000)*):
    - are vague in their descriptions of the conceptual knowledge and problem solving abilities needed;
    - do not provide the level of guidance needed for mathematics faculty, instructors or secondary teachers to operationalize the general calls

    Question: What procedural knowledge and how relevant procedural knowledge be taught more meaningfully?

Some Areas in Which Consensus is Emerging

An instructor's content knowledge is critical for quality teaching

Problem: There is currently no consensus on the nature of the content knowledge needed for college teaching

Research supports: (e.g., Thompson (in press); Carlson & Oehrtman (in press), Ball, Ma, 200?): Deep and connected understand of the *concept being taught* is necessary but not sufficient for quality teaching.

Teachers also need: i) meaningful curriculum; ii) to attend to student thinking; iii) mechanisms for reflecting and modifying practice; iv) local support structures that support "quality" reflection on instructional practices; v) administrative support for experimentation

-Inservice for teachers and prof dev. can do better

Some Areas in Which Consensus is Emerging

- Formative assessments (at the classroom and departmental levels) are needed to evaluate the impact of local curriculum and instruction on student learning

  Problems: There is currently no consensus on what and how to assess student learning or effective teaching; no mechanisms exists in most local environments to support the generation of formative knowledge

  Research supports: Local communities engagement in reflection and adaptation of instructional practices have been shown to be productive for improving teaching practice—the leader of the professional learning community (PLC) is KEY to the quality of reflection!!!

Processes of Mathematics Education Research that Guide Instructional Practice

One Example

Design (Revise)

Practices & Instruments

Carlson & Bloom Problem solving framework

-Describes effective mathematical practices

Conceptual Frameworks (e.g., Fate of change, Function, Covariation, FTC)

-Characterizes understanding (e.g., reasoning abilities, connections, notation issues)

Theoretical grounding for

-Designing curricular modules

-Determining course structure

-Instrument development

Lens for researching emerging practices and understandings

Revise Cognitive Frameworks

A Problem Solving Study

Research Question

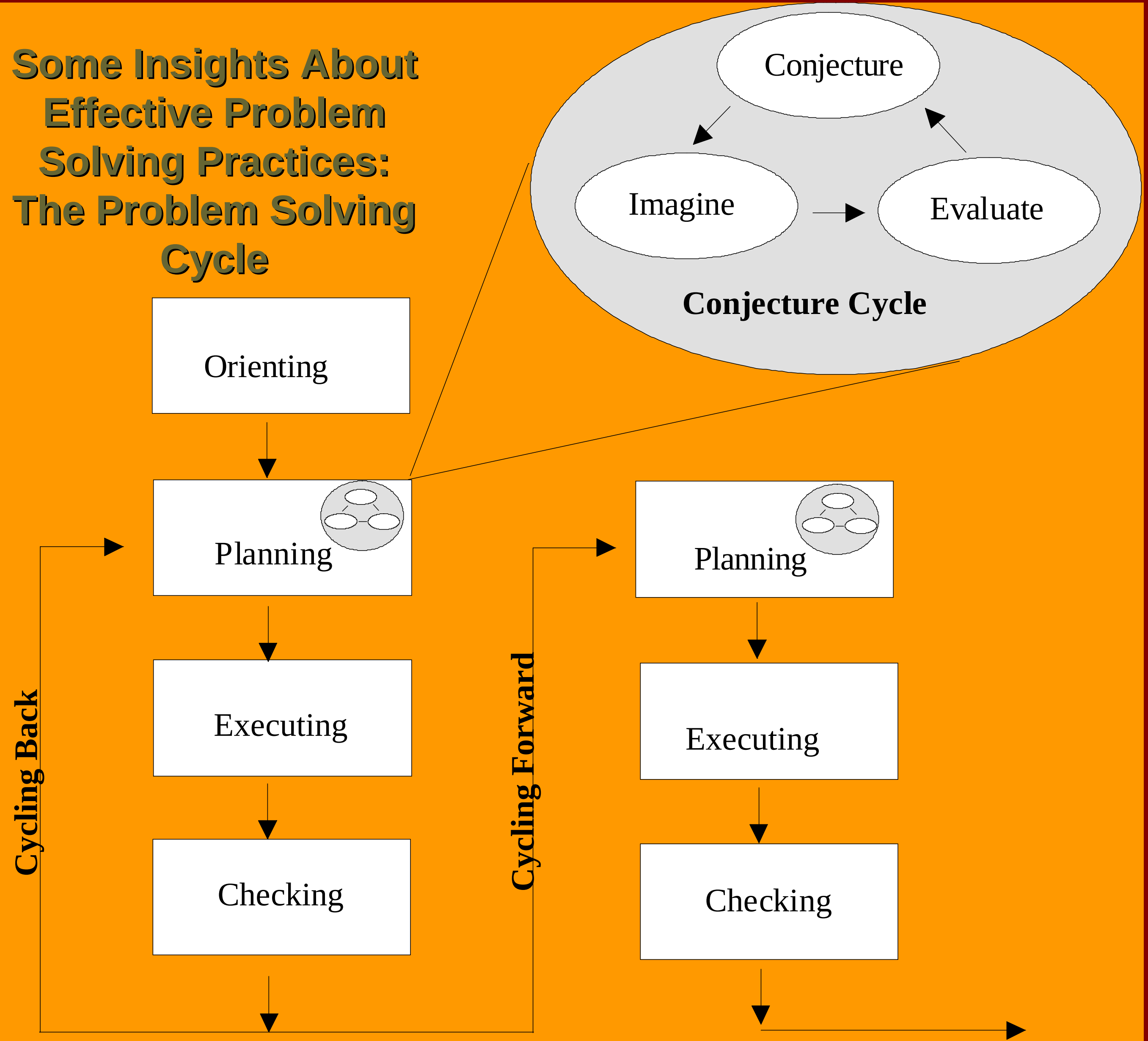
What are the abilities, knowledge, dispositions, etc. that contribute to success in problem solving?

- Methods:
- Clinical Interviews in which experts were asked to solve 5 novel (eighth grade level) problems
  - 18 graduate students (Carlson, 1999)
  - 12 mathematicians (Carlson & Bloom, 2003)

The Multidimensional Problem Solving Framework: A Characterization of Effective Problem Solving Behaviors

Phase •Behavior	Resources	Heuristics	Affect	Monitoring
Orienting •Sense making •Organizing •Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns, and classified the problem. (Solvers were sometimes observed saying, “this is an X kind of problem.”)	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking “What does this mean?”; “How should I represent this?”; “What does that look like?”
Planning •Conjecturing •Imagining •Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one’s abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, “Will this take me where I want to go?” and “How efficient will Approach X be?”
Executing •Computing •Constructing	Conceptual knowledge, facts, and algorithms were accessed when executing, computing, and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms, and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the reasonableness of the solution progress and products when constructing solution statements.
Checking •Verifying •Decision making	Resources, including well-connected conceptual knowledge, informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness, and aesthetic quality of the solution provided useful feedback to the solver.

Some Insights About Effective Problem Solving Practices: The Problem Solving Cycle



Other Select Findings

- Experts possess:
  - Powerful reasoning patterns
  - Deep and coherent understanding of fundamental concepts
  - Fluency in accessing and using facts, heuristics and procedures in novel setting at the right moment in advancing their solutions

Other Select Findings…

- Experts exhibited:
- Powerful metacognitive behaviors that lead to effective monitoring and decision making during problem solving
- persistence in solving novel problems
- Strong beliefs about the processes of doing mathematics
  - high confidence in their ability to reason through novel tasks
- Mechanisms for coping with frustration, pride
- Mathematical Integrity (intellectual honesty)
  - Conjectures were *always* based on a logical foundation
  - They never pretended to know when they didn't
- Mathematical Interest and Intimacy (????)

Carlson, M. & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem solving framework, *Educational Studies in Mathematics, 58*, 45-75.

Problem Solving Research Informs Classroom Practices

Rules of Engagement:

Individuals are expected to:

- Persist in making sense
  - Attempt to make logical connections
- Speak meaningfully—
  - Reference quantities and relationships—no use of pronouns!!
- Exhibit Mathematical Integrity
  - Don’t pretend to understand when you don’t
  - Conjectures should be based on a logical foundation
  - Respect the learning process of your peers
  - Ask questions—don’t tell

Homework—novel problems that require use of concepts

Example PCA Item

- A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the time, *t*, that has passed since the ball hit the water. (R2, C2)

Example PCA Item

- A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the time, *t*, that has passed since the ball hit the water. (R2, C2)
  - a) $A = 25\pi t$ *(21%)*
  - *b)* $A = \pi r^2$ (7%)
  - *c)* $A = 25\pi t^2$ (17%)
  - *d)* $A = 5\pi t^2$ (34%)
  - e) None of the above (20%)

Data from administering the exam to 652 precalculus students at the end of the course.

An example of reasoning that supported correct responses

I *imagined* the rock hitting the water and was able to *picture ripples traveling* outward.

<student illustrates rock and draws concentric circles with ripples>

Since the ripples *travel* out 5 cm *for every* second,

<student points to physical distance between ripples and says," the radius gets 5 cm longer every second">

This means the *length* of the radius, r, is 5 times the number of seconds that have passed. Since I need the area and I know $A = \pi r^2$ then $A = \pi(5t)^2$ When I square the radius I get $25\pi t^2$ so the size of the circle—I mean the area of the circle—grows to $25\pi t^2$

So, if 3 seconds *elapses*, the radius *grows* to 15 cm; then that area is 225 π square cm. That makes sense.

The Falling Ladder Problem

From a vertical position against a wall, the bottom of a ladder is pulled away at a constant rate. Describe the speed of the top of the ladder as it slides down the wall.

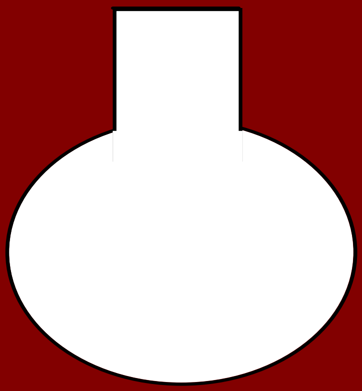
(Carlson, 1998)

Meaning of sine function

- See applet

The Bottle Problem

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle



Kim’s Response

As more water comes into the bottle, the height of the water in the bottle will rise.

Savanna’s response

I imagined equal amounts of water and placed a point on the graph of how high it would go up..then I imagined putting in the same amount of water and saw that it would go up less high on the bottle until it reached the widest point of the bottle…then as I keep adding the same amount of water and it would go up higher and higher until it reaches the neck.

Mental Actions of the Covariational Reasoning Framework

MA1) ***Coordinating one variable with changes in the other variable.***

MA2) ***Coordinating the direction of change in one variable with changes in the other variable*** **(e.g., increasing, decreasing);**

MA3) ***Coordinating the amount of change of one variable with changes in the other variable.***

MA4) ***Coodinating the average rate of change of one variable (with respect to the other variable) with uniform changes in the other variable.***

MA5) ***Coordinating the instantaneous rate of change of one variable (with respect to the other variable) with continuous changes in the other variable.***

(Carlson, Jacobs, Coe, Larsen, Hsu, 2002)

Covariational Reasoning

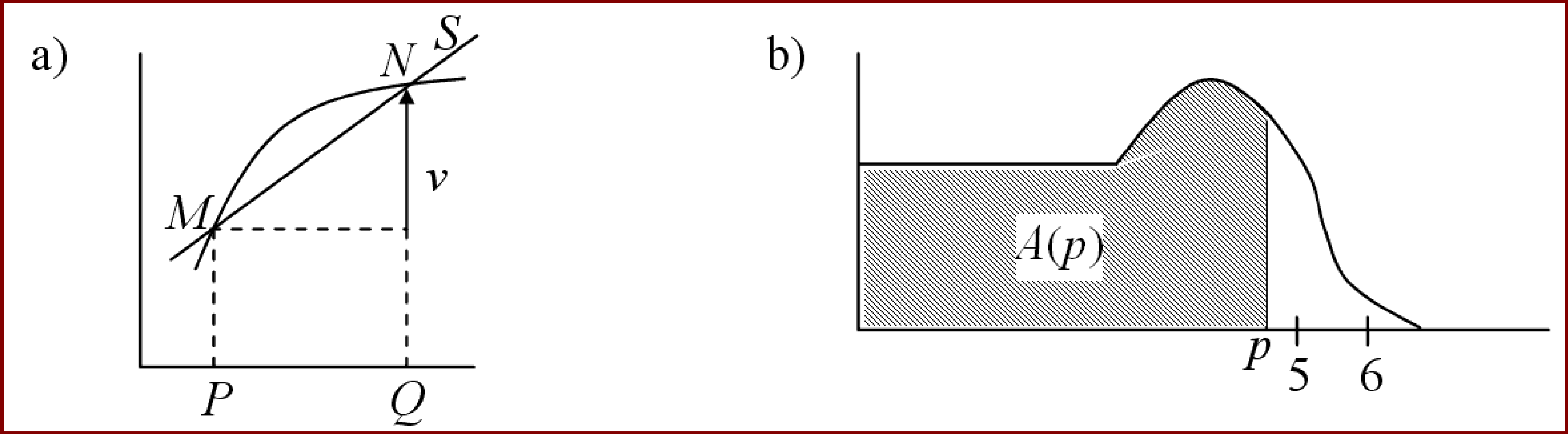
Claim:

Covariational Reasoning is a "way of thinking" that is foundational for understanding many of the major concepts of calculus.

Covariational Reasoning:
A Foundational Reasoning Ability for:

See Oehrtman, Carlson & Thompson (2009)

Derivative   Accumulation



$$A(p)=\int_0^p f(t)\,dt$$

The Role of Mathematics Education Frameworks

Cognitive models of knowing and learning, such as the *Covariation Framework* and the *Multidimensional Problem Solving Framework* increase the purposefulness of course design, instructional materials, instructional methods, assessment tools, and serve as guiding frameworks for evaluating student thinking.

Concepts and Reasoning in Calculus

- The ideas of rate of change, covariation and function are critical to student success in calculus (e.g., Carlson, 1995, 1997, 1998, 2002; Cotrill; 1996; Oehrtman, Carlson, Thompson, 1992, 1994, 2002, 2008; Dubinsky, 1992, 1995; Kaput, 1992; Monk, 1989; Rasmussen, 1998; Sierpinska, 1992; Zandieh, 2000).
- Quantitative and covariational are essential ways of thinking that promote:
  - Connections across function representations
  - Understanding key ideas of precalculus and calculus
    - Related Rate Problems (Engelke, 2006)
    - Exponential Functions (Strom, 2007)
    - Fundamental Theorem of Calculus (Smith, 2008)
    - Angle Measure as an arc length; Sine and Cosine Functions (Moore, in preparation)
    - Function composition and inverse (Bowling, in preparation)

The Precalculus Concept Assessment Instrument

- A 25 item multiple choice exam (80 Item Pool)
- Based on the body of literature related to knowing and learning the concept of function (e.g., Carlson, 1998, 1999, 2003; Dubinsky et al., 1992; Monk,1992; Thompson and Thompson, 1992; Thompson, 1994; Sierpinska, 1992)
- Validated and refined over a 12 year period

PCA Taxonomy

- ***Reasoning Abilities***
  - R1: Apply proportional reasoning
  - R2: View a function as a process that accepts input and produces output
  - R3: Apply covariational reasoning
    - Coordinate two varying quantities while attending to how the quantities change in relation to each other

PCA Taxonomy (cont.)

- ***Conceptual Abilities***
  - C1: Evaluate and interpret function information
  - C2: Represent contextual function situations using function notation
  - C3: Understand and perform function operations
  - C4: Understand how to reverse the function process
  - C5: Interpret and represent function behaviors
  - C6: Interpret and represent rate of change information for a function

PCA website

http://mathed.asu.edu/instruments/PCA/index.shtml

The Precalculus Concept Assessment: A Tool for Assessing Reasoning Abilities and Understandings of Precalculus Level Students (Carlson, Oehrtman and Engelke, to appear)

Concluding Remarks

- Mathematics education research has much to offer about:
  - Processes of reflecting on and adapting practices that are based on scientific methods
  - how to scaffold mathematics instruction and curriculum so that it is more meaningful for students

Many studies exist regarding how specific ideas and foundational reasoning abilities develop; approaches to teaching proof; PDP workshop model and why it works, differential equations, insights on preparing graduate student to teach, etc.

See MAA Notes #73: Research to Practice: Making the Connection: Research and Teaching in Undergraduate Mathematics Education (Carlson & Rasmussen, 2008)

The Precalculus Concept Assessment Instrument (PCA)

- A broad taxonomy identifies specific concepts and what is involved in understanding those concepts
  - Based on broad body of research on knowing and learning those concepts
- Multiple items assess each taxonomy attribute
- Items have multiple choice responses
- Item choices are based on common responses that have been identified in students (clinical interviews)
- Videos of student reasoning are captured for each item choice

Average Velocity Item

The distance, s (in feet), traveled by a car moving in a straight line is given by the function, $s(t)=t^2+t$, where t is measured in seconds. Find the average velocity for the time period from $t = 1$ to $t = 4$.

(11% of 652 precalculus students provided the correct response)

- **Reasoning Ability:** *R2* View a function as a process
- **Conceptual Ability:** *C6* Determine average rate-of-change information given a formula.

Student PCA Performance

- PCA administered to 550 college algebra and 379 pre-calculus students at a large southwestern university
- Mean score for end-of-course college algebra: 6.8/25
- Mean score for end-of-course pre-calculus: 9.1/25
- Also administered to 267 pre-calculus students at a nearby community college
- N=1205 for item data

Secondary Mathematics Teachers' PCA Scores

- Mean Score: 17/25 before intervention
- Mean Score: 23/25 after intervention

Predictive Potential of PCA

- How does PCA score relate to course grade in calculus?
  - Tracking 277 students in beginning calculus, 81% of students who scored 13 or above on PCA at the beginning of the semester received an A, B, or C in calculus.
  - 85% of the 277 students who scored 11 or below received a D, F or withdrew.

Uses of PCA

- Provides coherent assessment of central ideas and what is involved in understanding those ideas
- Provides valid and reliable assessment of student understanding
- Easy to administer to large populations
- Can assess course and instructional effectiveness
- Website exists with videos of correct and incorrect reasoning can promote reflection on student thinking and learning
- May serve as a tool to assess readiness for calculus

Limitations of Concept Assessment Instruments

- Do not reveal thinking of individual students
- Do not reveal insights into the process of learning
- Are not tied to specific instruction or course materials
- Limited in assessing creative abilities

Common reasoning supporting the answer $A(t) = 5\pi\, t^2$

I know the time would go in for radius so that would give me $A = \pi t^2$ and since the circle grows at the rate of 5 cm per second I would get $A(t) = 5\pi\, t^2$

Reasoning that supported correct response

I *imagined* the rock hitting the water and was able to *picture ripples traveling* outward.

<student illustrates rock and draws concentric circles with ripples>

Since the ripples *travel* out 5 cm *for every* second,

<student points to physical distance between ripples and says," the radius gets 5 cm longer every second">

This means the *length* of the radius, r, is 5 times the number of seconds that have passed. Since I need the area and I know $A = \pi r^2$ then $A = \pi(5t)^2$ When I square the radius I get $25 \pi t^2$ so the size of the circle—I mean the area of the circle—grows to $25 \pi t^2$

So, if 3 seconds *elapses*, the radius *grows* to 15 cm; then that area is 225 π square cm. That makes sense.

What did you notice about this student's reasoning?

- She was imagining:
  - Objects in a system (rock, radius, time, area, water)
  - Some attributes of objects in the system and how they are related (e.g., length of the radius in cm)
  - How the amount of time and length of the radius changed together
  - The quantities in the system varying together: A dynamical system of quantities
  - How the size of the circle (area) changed with the change in the time (loose association)

Conceptualizing a Quantity:

A person is thinking of a quantity when he or she conceives of an attribute of object in such a way that this conception entails the attributes measurability

Quantitative operation:

The conception of two quantities being taken to produce a new quantity

(Thompson, 1994; Moore & Carlson, in preparation)

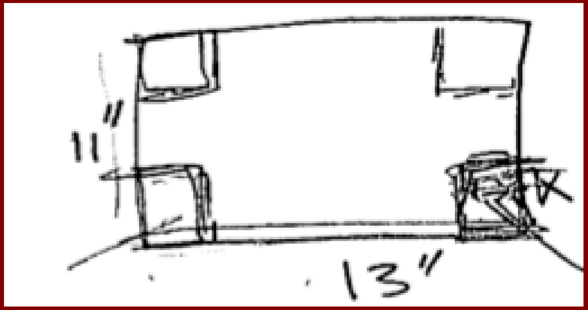
Instructional Approaches to Promote Development of Skills and Understandings

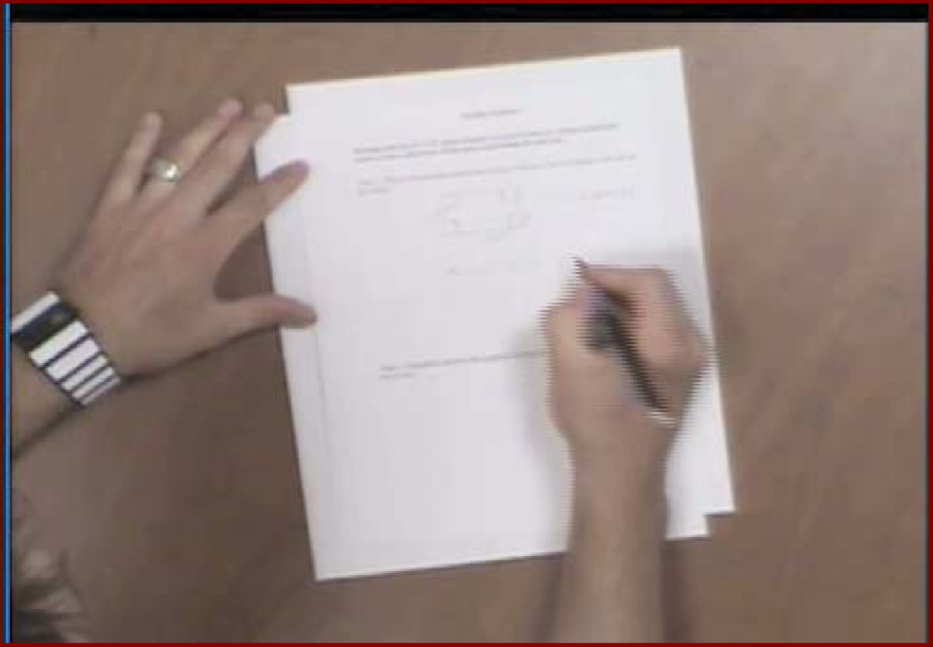
The Box Problem

Starting with an 11”x13” sheet of paper a box is formed by cutting equal sized squares from each corner and folding up the sides. Write a formula that predicts the volume of the box from the length of the side of the cutout.

The Box Problem

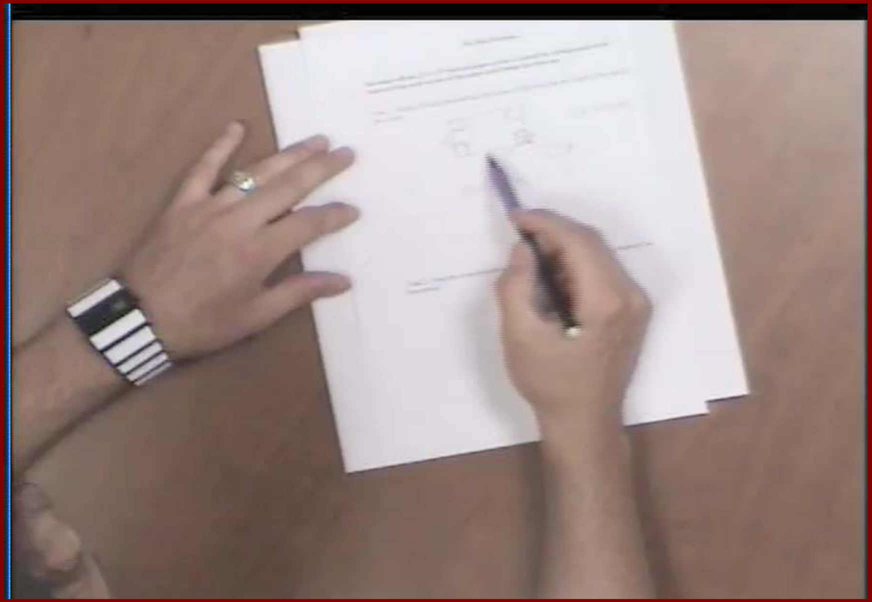
- Occurred after he had created a diagram and determined *V*=13*11**x*





- Travis appeared to not conceive of the measurement process of the length and width of the box.
- The length and the width of the box had fixed measurements.

The Box Problem



- Occurred after being asked about the side of the box.

- Travis appeared to conceive of the various quantities of the situation and how these were related.
- The symbolic notation of *x* represented a *number* of inches of the side of the cutout.

The Box Problem

Planting Seeds

The Box Problem

Planting Seeds

Results from select research studies

- Passing rates in first-semester calculus by students who had passed precalculus range from 17%-47 at 6 univ. (Schattschneider's, 2006; cited in Bressoud, 2009).

- 54% of students who passed beginning calculus with a C or better (at a large public university in the SW) and had declared a major that required calculus II either did not enroll in or did not pass second semester calculus (Thompson, 2007).

- Students are not acquiring the conceptual knowledge, problem solving abilities, or foundational reasoning abilities needed for continuing in mathematics course taking (e.g., Carlson, 1998, 2002, 2003, 2007; Carlson & Rasmussen, 2008)

Analysis of each facilitator revealed four observable manifestations of decentering, with a fifth manifestation (FDM5) following our theoretical perspective. We have characterized these Facilitator Decentering Moves (FDMs) as follows.

FDM1: The facilitator shows no interest in understanding the thinking or perspective of a PLC member with which he/she is interacting.

FDM2: The facilitator takes actions to model a PLC member's thinking, but does not use that model in communication with the PLC member.

FDM3: The facilitator builds a model of a PLC member's thinking and recognizes that it is different from her/his own. The facilitator then acts in ways to move the PLC member to her/his way of thinking, but does so in a manner that does not build on the rationale of the other member.

FDM4: The facilitator builds a model of a PLC member's thinking and acts in ways that respect and build on the rationality of this member's thinking for the purpose of advancing the PLC member's thinking and/or understanding.

FDM5: The facilitator builds a model of a PLC member's thinking and respects that it has a rationality of its own. Through interaction, the facilitator also builds a model of how he/she is being interpreted by the PLC member. He/she then adjusts her/his actions (questions, drawings, statements) to take into account both the PLC member's thinking and how the facilitator might be interpreted by that PLC member.