# The universal $sl(2)$ foam cohomology

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Homology Theories of Knots and Links
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# The plan

- We have constructed a bi-graded cohomology theory for tangles that depends on two parameters $h$ and $a$, which for the case of links, is a categorification of the classical Jones polynomial.

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- We have constructed a bi-graded cohomology theory for tangles that depends on two parameters $h$ and $a$, which for the case of links, is a categorification of the classical Jones polynomial.
- This cohomology theory is properly functorial under link cobordisms, relative to boundary.
- For certain values of parameters $a, h$ we obtain isomorphic versions of the Khovanov homology and Lee's variant of it.
- Therefore, we also obtain for free a version of the Khovanov homology which satisfies functoriality property with no sign ambiguity.

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- Our construction is a geometric approach to this universal homology theory.

We need the condition that the ground ring contains the fourth-root of unity $i$. We also work with cobordisms with seams, as opposed to ordinary cobordisms.

- Let $R := \mathbb{Z}[i][a, h]$.
- We introduce a grading on the ring $R$:

$$\deg(h) = 2, \deg(a) = 4$$

$$\deg(1) = \deg(i) = 0.$$

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- Let $\mathcal{A} = R[X]/(X^2 - hX - a)$ be the free $R$-module with generators $\mathbf{1}$ and $\mathbf{X}$.
- We introduce a grading on $\mathcal{A}$ by letting:

$$\deg(\mathbf{1}) = -1, \deg(\mathbf{X}) = 1.$$

$\mathcal{A}$ is commutative Frobenius with:

- multiplication

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \begin{cases} m(\mathbf{1} \otimes \mathbf{X}) = \mathbf{X} & m(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \\ m(\mathbf{X} \otimes \mathbf{1}) = \mathbf{X} & m(\mathbf{X} \otimes \mathbf{X}) = h\mathbf{X} + a \end{cases}$$

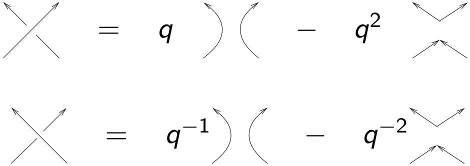
- comultiplication

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \begin{cases} \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{X} + \mathbf{X} \otimes \mathbf{1} - h\mathbf{1} \otimes \mathbf{1} \\ \Delta(\mathbf{X}) = \mathbf{X} \otimes \mathbf{X} + a\mathbf{1} \otimes \mathbf{1} \end{cases}$$

- unit $\iota : R \to \mathcal{A}, \quad \iota(1) = \mathbf{1}$
- counit $\epsilon : \mathcal{A} \to R, \epsilon(\mathbf{1}) = 0, \epsilon(\mathbf{X}) = 1.$

# Computing the quantum $sl(2)$-link invariant via webs

- The quantum $sl(2)$-link invariant is determined by the rules:



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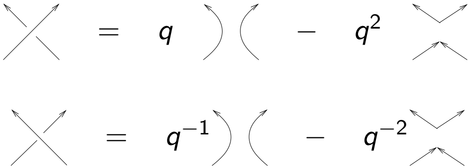
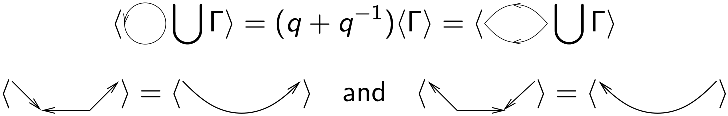



- This sends an oriented link diagram to a $\mathbb{Z}[q, q^{-1}]$-linear combination of disjoint union of *webs*.
- Webs are circle graphs with bivalent vertices such that edges incident to a vertex are either both entering or both leaving the vertex.

# The web space

- Webs are evaluated via the relations:



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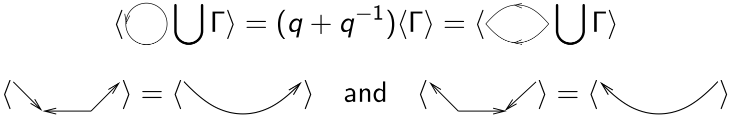
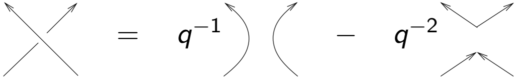


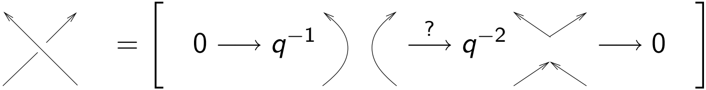

- By these rules, a web with $k$ connected components evaluates to $(q+q^{-1})^k$.
- The web space gives another way of computing the $sl(2)$-link invariant.

# A method for categorification

To categorify the $sl(2)$-link invariant we want to associate complexes to links instead of linear combinations.
Thus we want to replace a linear combination like

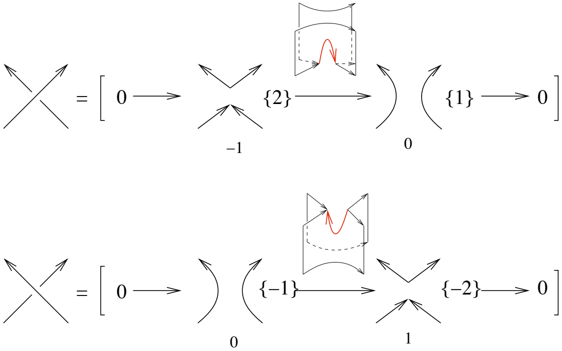



with some complex of the form



# Constructing complexes from link diagrams

Starting with a generic link diagram $D$, we associate to it a formal complex $[D]$ whose construction is explained by:



## Constructing complexes from link diagrams

- The “chain objects” of $[D]$ are column vectors of *webs* and “differentials” are matrices of *foams*.
- A foam is a cobordism between webs, and contains seams where orientations disagree.
  Specifically, the two facets incident with a seam have opposite orientations, which induces the same orientation on that seam.

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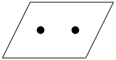
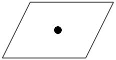
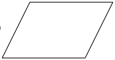
- A foam is a cobordism between webs, and contains seams where orientations disagree.
  Specifically, the two facets incident with a seam have opposite orientations, which induces the same orientation on that seam.
- There is an ordering of the facets sharing a seam.
- Foams can have dots that are allowed to move freely along a facet they belong to, but can't cross seams.
- We read foams (as morphisms) from bottom to top, by convention, and we compose them by placing one on top of the other.

# The category **Foams**

- Let **Foams** be the category whose objects are webs and whose morphisms are foams, regarded up to boundary-preserving isotopies.
- We introduce a set of relations $\ell$ among foams and denote the corresponding category by $\mathbf{Foams}_{/\ell}$.
- These relations are necessary in order to obtain a complex that is a link invariant (already) at the geometric level.

# Local relations

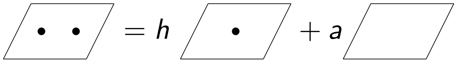
(2$D$)  $= h$  $+ a$ 

(2D) is the geometric counterpart of relation in $\mathcal{A}$:

$$X^2 - hX - a = 0$$

# Local relations

(2D)

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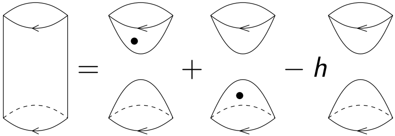
($CN$)

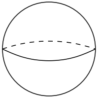
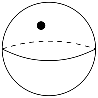



The surgery formula (CN) is the geometric counterpart of:

$$\Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1$$

# Local relations

## Closed foam relations

(S)  $= 0$,  $= 1$

These relations correspond to $\epsilon(1) = 0$, and $\epsilon(X) = 1$, respectively.

# Local relations

## Closed foam relations

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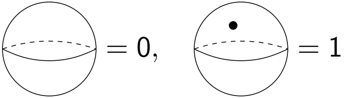


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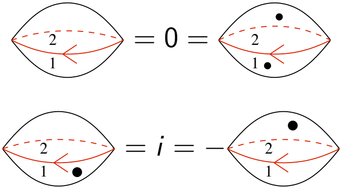
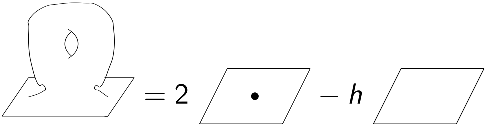
(F)



- The cutting neck relation (CN) implies the genus reduction formula:



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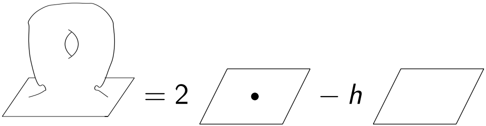
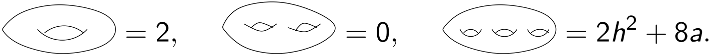
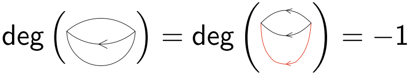


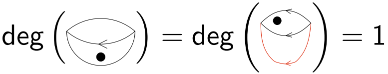

- By using the local relations $\ell$, we assign to any closed foam a unique element in $R$.
  In particular we have:



## Both categories **Foams** and **Foams**$_{/\ell}$ are graded by degree

- We introduce a grading on the category **Foams** by defining the degree of a foam $F$ with $d$ dots: $\deg(F) = -\chi(S) + 2d$, where $\chi$ is the Euler characteristic.





- The local relations $\ell$ are degree-preserving, so **Foams**$_{/\ell}$ is also graded.
- Differentials in $[D]$ are degree-preserving.

# Invariance and functoriality

**Theorem**

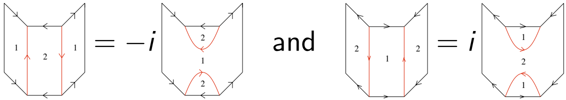
*The formal complex $[D]$ is invariant under the Reidemeister moves, up to homotopy.*

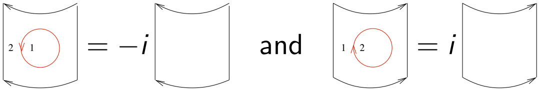
**Theorem**

*This theory is properly functorial under link cobordisms; that is, given a 4-dimensional cobordism $C$ between two links $L_1$ and $L_2$, there is a well defined homomorphism $[C] : [L_1] \rightarrow [L_2]$ between the corresponding formal complexes.*

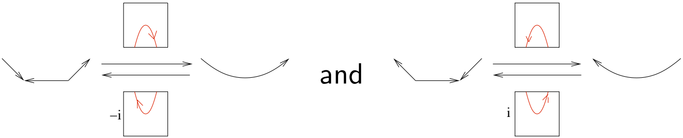
# Some isomorphisms

The following relations hold in the category **Foams**$_{/\ell}$:




which imply the isomorphisms below:



# Passing to an algebraic category

We are now ready to apply a degree-preserving functor $\mathcal{F}$ to pass to an algebraic category, and obtain an ordinary complex $\mathcal{F}([D])$:

$$\mathbf{Foams}_{/\ell} \xrightarrow{\mathcal{F}} R - \mathbf{Mod}$$

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**Theorem**

*The isomorphism class of the cohomology $\mathcal{H}(D) := H(\mathcal{F}([D]))$ is a bi-graded invariant of L, whose graded Euler characteristic is the $sl(2)$ invariant of L.*

# How is the functor $\mathcal{F}$ defined?

- Let $\emptyset$ be the empty web.
- Define the functor $\mathcal{F}$ : $\mathbf{Foams}_{/\ell} \longrightarrow R\text{-}\mathbf{Mod}$ as follows:

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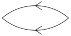
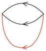
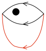
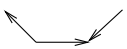
on morphisms: Given $S : \Gamma' \to \Gamma''$ define

$$\begin{array}{rccc} \mathcal{F}(S): & \mathrm{Mor}(\emptyset, \Gamma') & \to & \mathrm{Mor}(\emptyset, \Gamma'') \text{ by} \\ & U & \longmapsto & S \circ U \end{array}$$

## $\mathcal{F}(\Gamma)$ mimics the web space skein relations

We have:

- $\mathcal{F}(\emptyset) = R$
- $\mathcal{F}($  $) = \langle$  ,  $\rangle_R \cong \mathcal{A}$
- $\mathcal{F}($  $) = \langle$  ,  $\rangle_R \cong \mathcal{A}$
- $\mathcal{F}(\Gamma \cup$  $) \cong \mathcal{F}(\Gamma) \otimes_R \mathcal{A} \cong \mathcal{F}(\Gamma \cup$  $)$
- $\mathcal{F}($  $) \cong \mathcal{F}($  $)$, $\mathcal{F}($  $) \cong \mathcal{F}($  $)$

Therefore $\mathcal{F}(\Gamma)$ is a free abelian group of graded rank $\langle \Gamma \rangle$.

# The invariant of a surface-knot

- Our theory is functorial under link cobordisms.
- A surface-knot (or surface-link) $S$ is a closed surface embedded locally flatly in $\mathbb{R}^4$, and can be regarded as a cobordism from the empty link to itself.

# The invariant of a surface-knot

- Our theory is functorial under link cobordisms.
- A surface-knot (or surface-link) $S$ is a closed surface embedded locally flatly in $\mathbb{R}^4$, and can be regarded as a cobordism from the empty link to itself.
- The induced map $\mathcal{H}(\emptyset) \to \mathcal{H}(\emptyset)$ is an endomorphism of the ground ring $R = \mathbb{Z}[i][a, h]$, and yields an invariant of $S$, denoted here by $\mathcal{L}(S)$.
- The invariant $\mathcal{L}(S)$ is determined only by the genus of $S$.

# The invariant of a surface-knot

**Theorem**

*For any surface-knot $S$ of genus $g$, we have the following:*

1. *If $g$ is even, then $\mathcal{L}(S) = 0$;*
2. *If $g$ is odd, then $\mathcal{L}(S) = 2(h^2 + 4a)^{\frac{g-1}{2}}$.*

**Corollary**

*For any torus-knot $T^2$, we have $\mathcal{L}(T^2) = 2$.*

# Working over $\mathbb{C}$

- Now let $a, h \in \mathbb{C}$ and work over $\mathbb{C}$.
- There are two isomorphism classes of the corresponding link cohomology over $\mathbb{C}$, which are determined by the number of distinct roots of $f(X) = X^2 - hX - a$.

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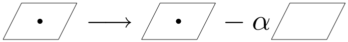
- There are two isomorphism classes of the corresponding link cohomology over $\mathbb{C}$, which are determined by the number of distinct roots of $f(X) = X^2 - hX - a$.

(1) If $h^2 + 4a = 0$, thus $f(X) = (X - \alpha)^2$ for some $\alpha \in \mathbb{C}$, there is an isomorphism between $\mathcal{H}_{a,h}(L, \mathbb{C})$ and the original Khovanov homology over $\mathbb{C}$, induced by the isomorphism:



# Working over $\mathbb{C}$

(2) If $f(X) = (X-\alpha)(X-\beta)$, for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$, we have:

**Theorem**

*For any link with n components, $dim_{\mathbb{C}}\mathcal{H}_{a,h}(L, \mathbb{C}) = 2^n$, and to each map $\psi : \{\text{components of } L\} \longrightarrow \mathcal{S} = \{\alpha, \beta\}$, there exists a non-zero element $\exists h_\psi \in \mathcal{H}_{a,h}(L, \mathbb{C})$ which lies in the cohomological degree*

$$-2 \sum_{(u_1,u_2)\in \mathcal{S}\times\mathcal{S}, u_1\neq u_2} lk(\psi^{-1}(u_1), \psi^{-1}(u_2)).$$

*All $h_\psi$'s generate $\mathcal{H}(L, \mathbb{C})$.*

# Filtered theory

Considering $a, h \in \mathbb{C}$ such that $h^2 + 4a \neq 0$, one looses the bi-grading and obtains a $\mathbb{Z}$-graded theory with a filtration in place of the second grading (polynomial grading).

**Theorem**

*For each $a, h \in \mathbb{C}$ such that $h^2 + 4a \neq 0$, there is a spectral sequence converging to $\mathcal{H}_{a,h}(L, \mathbb{C})$ with $E_1$ term isomorphic to Khovanov's invariant over $\mathbb{C}$.*

**Theorem**

*The $E_1$ and higher terms of this spectral sequence are link invariants.*

# Bibliography


- C., **The universal $sl(2)$ cohomology via webs and foams**, *Topology and Its Applications* **156** (2009), 1684–1702; arXiv:math.GT/0802.2848,
- C., **On the $sl(2)$ foam cohomology computations**, *J. Knot Theory Ramifications* **18** Issue 9 (2009), 1313-1328; arXiv:math.GT/0805.4651.


THANK YOU!