Khovanov-Rozansky homology

Ben Webster

MIT/Oregon

January 28, 2010

References:

This slide show can be downloaded from

http://math.mit.edu/~bwebster/berkeley-2.pdf

$\mathfrak{gl}_N$-invariants

So we've already talked about knot Floer homology, Khovanov homology and HOMFLYPT homology. These actually fit into a larger picture.

Polynomial	Homology	Lie (super)algebra
Alexander	$HFK(-)$	$\mathfrak{gl}(1\|1)$
$2^{\#\text{components}}$	Lee	$\mathfrak{gl}(1)$
Jones	$Kh(-)$	$\mathfrak{gl}(2)$
??	??	??
HOMFLYPT	$HHH(-)$	$\mathfrak{gl}(\infty)$

$\mathfrak{gl}_N$-invariants

So we've already talked about knot Floer homology, Khovanov homology and HOMFLYPT homology. These actually fit into a larger picture.

Polynomial	Homology	Lie (super)algebra
Alexander	$HFK(-)$	$\mathfrak{gl}(1\|1)$
$2^{\#\text{components}}$	Lee	$\mathfrak{gl}(1)$
Jones	$Kh(-)$	$\mathfrak{gl}(2)$
$\mathfrak{gl}_N$-invariants	??	$\mathfrak{gl}(N)$
HOMFLYPT	$HHH(-)$	$\mathfrak{gl}(\infty)$

$\mathfrak{gl}_N$-invariants

So we've already talked about knot Floer homology, Khovanov homology and HOMFLYPT homology. These actually fit into a larger picture.

Polynomial	Homology	Lie (super)algebra
Alexander	$HFK(-)$	$\mathfrak{gl}(1\|1)$
$2^{\#\text{components}}$	Lee	$\mathfrak{gl}(1)$
Jones	$Kh(-)$	$\mathfrak{gl}(2)$
$\mathfrak{gl}_N$-invariants	Khovanov-Rozansky	$\mathfrak{gl}(N)$
HOMFLYPT	$HHH(-)$	$\mathfrak{gl}(\infty)$

$\mathfrak{gl}_N$-invariants

So we've already talked about knot Floer homology, Khovanov homology and HOMFLYPT homology. These actually fit into a larger picture.

Polynomial	Homology	Lie (super)algebra
Alexander	$HFK(-)$	$\mathfrak{gl}(m\|m)$
$2^{\#\text{components}}$	Lee	$\mathfrak{gl}(m+1\|m)$
Jones	$Kh(-)$	$\mathfrak{gl}(m+2\|m)$
$\mathfrak{gl}_N$-invariants	Khovanov-Rozansky	$\mathfrak{gl}(m+N\|m)$
HOMFLYPT	$HHH(-)$	$\mathfrak{gl}(\infty)$

$\mathfrak{gl}_N$-invariants

These are easily defined as specialzations of the HOMFLY polynomial

$$P_n(q) = P(q, q^n).$$

Now, you might be tempted do say "Well, that's easy. Just specialize the gradings." That's appealing, because it's simple, but there are good reasons not to.

$\mathfrak{gl}_N$-invariants

These are easily defined as specialzations of the HOMFLY polynomial

$$P_n(q) = P(q, q^n).$$

Now, you might be tempted do say "Well, that's easy. Just specialize the gradings." That's appealing, because it's simple, but there are good reasons not to.

- HOMFLYPT homology is infinite dimensional; there's a fix to make it finite dimensional for knots, but for links there's no hope. K-R homology is always *finite dimensional*.

$\mathfrak{gl}_N$-invariants

These are easily defined as specialzations of the HOMFLY polynomial

$$P_n(q) = P(q, q^n).$$

Now, you might be tempted do say "Well, that's easy. Just specialize the gradings." That's appealing, because it's simple, but there are good reasons not to.

- HOMFLYPT homology is infinite dimensional; there's a fix to make it finite dimensional for knots, but for links there's no hope. K-R homology is always *finite dimensional*.
- HOMFLYPT homology isn't (as far as we know) functorial. K-R homology is *functorial* (up to sign?).

$\mathfrak{gl}_N$-invariants

These are easily defined as specialzations of the HOMFLY polynomial

$$P_n(q) = P(q, q^n).$$

Now, you might be tempted do say "Well, that's easy. Just specialize the gradings." That's appealing, because it's simple, but there are good reasons not to.

- HOMFLYPT homology is infinite dimensional; there's a fix to make it finite dimensional for knots, but for links there's no hope. K-R homology is always *finite dimensional*.
- HOMFLYPT homology isn't (as far as we know) functorial. K-R homology is *functorial* (up to sign?).
- Having more knot homologies is having more information. K-R homology is one more thing to compare, and there are also many spectral sequences one can compare.

Khovanov-Rozansky homology

So, Khovanov-Rozansky homology is a sort of interpolation between Khovanov and HOMFLY homology. Interestingly, it has several different faces:

Khovanov-Rozansky homology

So, Khovanov-Rozansky homology is a sort of interpolation between Khovanov and HOMFLY homology. Interestingly, it has several different faces:

- The original definition is due to Khovanov and Rozansky, and very similar to the description that Scott gave. One defines functors between categories of matrix factorizations associated tangles, and then takes the functor from $\mathsf{Vect} \to \mathsf{Vect}$ given by a knot.

Khovanov-Rozansky homology

So, Khovanov-Rozansky homology is a sort of interpolation between Khovanov and HOMFLY homology. Interestingly, it has several different faces:

- The original definition is due to Khovanov and Rozansky, and very similar to the description that Scott gave. One defines functors between categories of matrix factorizations associated tangles, and then takes the functor from $\mathsf{Vect} \to \mathsf{Vect}$ given by a knot.
- A definition similar to that of HOMFLY homology; it's in some sense a "deformation" of this definition.

Khovanov-Rozansky homology

So, Khovanov-Rozansky homology is a sort of interpolation between Khovanov and HOMFLY homology. Interestingly, it has several different faces:

- The original definition is due to Khovanov and Rozansky, and very similar to the description that Scott gave. One defines functors between categories of matrix factorizations associated tangles, and then takes the functor from $\mathsf{Vect} \to \mathsf{Vect}$ given by a knot.
- A definition similar to that of HOMFLY homology; it's in some sense a "deformation" of this definition.
- There is a second discovered later by Mackaay, Stosic and Vaz, more reminiscent of the original Khovanov homology. I'm afraid we won't have time for those.

Matrix factorizations

Khovanov and Rozansky's original definition use matrix factorizations. Fix a commutative ring R and a scalar $w \in R$.

A **matrix factorization of potential** w is a 2-periodic complex R-modules

$$N_0 \underset{\partial}{\overset{\partial}{\rightleftarrows}} N_1$$

such that $\partial^2 = w$.

The name comes from the fact that this writes the scalar matrix wI as the product of two other matrices.

Koszul matrix factorizations

One cool fact about matrix factorizations (which doesn't work at all for usual factorizations) is that one can add the quantities being factored by taking tensor product: given MFs **M** and **N**, we have a new MF of potential $w_M + w_N$:

$$\mathbf{M} \otimes \mathbf{N} = \begin{array}{c} M_0 \otimes N_0 \\ \oplus \\ M_1 \otimes N_1 \end{array} \underset{\partial_M \otimes 1 + 1 \otimes \partial_N}{\overset{\partial_M \otimes 1 + 1 \otimes \partial_N}{\rightleftarrows}} \begin{array}{c} M_0 \otimes N_1 \\ \oplus \\ M_1 \otimes N_0 \end{array}$$

Koszul matrix factorizations

One cool fact about matrix factorizations (which doesn't work at all for usual factorizations) is that one can add the quantities being factored by taking tensor product: given MFs **M** and **N**, we have a new MF of potential $w_M + w_N$:

$$\mathbf{M} \otimes \mathbf{N} = \begin{array}{c} M_0 \otimes N_0 \\ \oplus \\ M_1 \otimes N_1 \end{array} \underset{\partial_M \otimes 1 + 1 \otimes \partial_N}{\overset{\partial_M \otimes 1 + 1 \otimes \partial_N}{\rightleftarrows}} \begin{array}{c} M_0 \otimes N_1 \\ \oplus \\ M_1 \otimes N_0 \end{array}$$

So for an expression $w = \sum a_i b_i$, we get a matrix factorizaton by tensoring together the smaller ones of the form

$$R \underset{b_i}{\overset{a_i}{\rightleftarrows}} R$$

Koszul matrix factorizations

If we start with the factorization

$$0 = \sum (x_i \otimes 1 - 1 \otimes x_i) \cdot 0$$

for the ring $R \otimes R$, this recipe gives the resolution of the diagonal bimodule we used before.

Koszul matrix factorizations

If we start with the factorization

$$0 = \sum (x_i \otimes 1 - 1 \otimes x_i) \cdot 0$$

for the ring $R \otimes R$, this recipe gives the resolution of the diagonal bimodule we used before.

So, we can *deform* this picture by changing the b_i's to something non-zero. Khovanov and Rozansky use

$$\frac{x_i^{N+1} \otimes 1 - 1 \otimes x_i^{N+1}}{x_i \otimes 1 - 1 \otimes x_i} = x_i^N \otimes 1 + \cdots + 1 \otimes x_i^N$$

So now we have a factorization $\mathbf{K}_N$ of potential $\sum_i x_i^{N+1} \otimes 1 - 1 \otimes x_i^{N+1}$.

Khovanov-Rozanky homology

Now, given a braid, we can take $F_i(\sigma) \otimes_{R \otimes R} \mathbf{K}_N$. This is a new matrix factorization over R, which is *killed by its potential*, since he potential is the difference of two symmetric polynomials.

Khovanov-Rozanky homology

Now, given a braid, we can take $F_i(\sigma) \otimes_{R\otimes R} \mathbf{K}_N$. This is a new matrix factorization over R, which is *killed by its potential*, since he potential is the difference of two symmetric polynomials.

When a matrix factorization is killed by its potential, we can take homology! Let $G_i(\sigma) = H^\bullet(F_i(\sigma) \otimes_{R\otimes R} \mathbf{K}_N)$ be the homology of this complex (it's actually all concentrated in degree 0).

Thus by functoriality, we obtain a complex:

$$\cdots \longrightarrow G_i(\sigma) \longrightarrow G_{i-1}(\sigma) \longrightarrow \cdots$$

Khovanov-Rozanky homology

The **Khovanov-Rozansky homology** of $\bar{\sigma}$ for $\mathfrak{sl}_N$ is the homology of this complex.

$$\cdots \longrightarrow G_i(\sigma) \longrightarrow G_{i-1}(\sigma) \longrightarrow \cdots$$

Theorem (Khovanov-Rozansky)

This is a doubly graded knot invariant which categorifies P_n. Furthermore, it is functorial for knot cobordisms.

When $N = 2$, this coincides with Khovanov homology.

Khovanov-Rozanky homology

There's an equivalent definition. We define a 2-category by

Khovanov-Rozanky homology

There's an equivalent definition. We define a 2-category by

- $0-\mathrm{Mor}=\mathbb{N}$.

Khovanov-Rozanky homology

There's an equivalent definition. We define a 2-category by

- $0-\mathrm{Mor} = \mathbb{N}$.
- $\mathrm{Hom}([n],[m]) =$ complexes of matrix factorizations over $\mathbb{C}[x_1,\cdots,x_n,y_1,\cdots,y_m]$ of potential $\sum_i x_i^{N+1} + \sum_j -y_j^{N+1}$.
- 2-morphisms are maps of complexes up to homotopy.

Khovanov-Rozanky homology

There's an equivalent definition. We define a 2-category by

- $0-\mathrm{Mor}=\mathbb{N}$.
- $\mathrm{Hom}([n],[m])=$ complexes of matrix factorizations over $\mathbb{C}[x_1,\cdots,x_n,y_1,\cdots,y_m]$ of potential $\sum_i x_i^{N+1}+\sum_j -y_j^{N+1}$.
- 2-morphisms are maps of complexes up to homotopy.

Tangles act on this as follows:

Khovanov-Rozanky homology

There's an equivalent definition. We define a 2-category by

- $0 - \mathrm{Mor} = \mathbb{N}$.
- $\mathrm{Hom}([n],[m]) =$ complexes of matrix factorizations over $\mathbb{C}[x_1,\cdots,x_n,y_1,\cdots,y_m]$ of potential $\sum_i x_i^{N+1} + \sum_j -y_j^{N+1}$.
- 2-morphisms are maps of complexes up to homotopy.

Tangles act on this as follows:

- a braid σ acts by $\otimes_R F(\sigma)$ as tensor product of complexes.

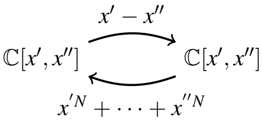
Khovanov-Rozanky homology

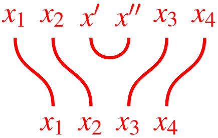
There's an equivalent definition. We define a 2-category by

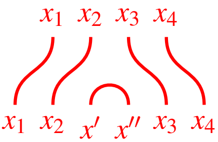
- $0 - \mathrm{Mor} = \mathbb{N}$.
- $\mathrm{Hom}([n],[m])$ = complexes of matrix factorizations over $\mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ of potential $\sum_i x_i^{N+1} + \sum_j -y_j^{N+1}$.
- 2-morphisms are maps of complexes up to homotopy.

Tangles act on this as follows:

- a braid σ acts by $\otimes_R F(\sigma)$ as tensor product of complexes.
- cup or cap is tensor product with the matrix factorization







Examples

Well, the first order of business is calculating the invariant of a circle:

$$R \otimes R \underset{x^N \otimes 1 + \cdots + 1 \otimes x^N}{\overset{x \otimes 1 - 1 \otimes x}{\rightleftarrows}} R \otimes R$$

Examples

Well, the first order of business is calculating the invariant of a circle:

$$R \xleftarrow[(N+1)x^N]{} R$$

Examples

Well, the first order of business is calculating the invariant of a circle:

$$H^*(\mathbb{CP}^N)$$

Functoriality

One easy hint that a knot invariant is functorial is that its invariant for a circle is a commutative Frobenius algebra, since this gives functoriality for cobordisms between unknots.

A **Frobenius algebra** over $\mathbb{C}$ is a $\mathbb{C}$-algebra, together with a map $t : A \to \mathbb{C}$ such that t is non-degenerate. That is,

- If $t(ab) = 0$ for all $a \Rightarrow b = 0$.

Functoriality

One easy hint that a knot invariant is functorial is that its invariant for a circle is a commutative Frobenius algebra, since this gives functoriality for cobordisms between unknots.

A **Frobenius algebra** over $\mathbb{C}$ is a $\mathbb{C}$-algebra, together with a map $t : A \to \mathbb{C}$ such that t is non-degenerate. That is,

- If $t(ab) = 0$ for all $a \Rightarrow b = 0$.

The cohomology of any compact manifold is commutative Frobenius in a canonical way:

- t is integration of top level cochains.
- Non-degeracy is equivalent to Poincaré duality.

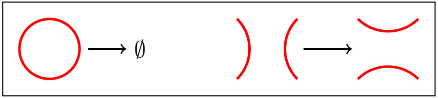
Functoriality

Thus $H^*(\mathbb{CP}^{N-1})$ is a good answer for the circle. $\mathbb{C}[x] \otimes \wedge(y)$, not so much.

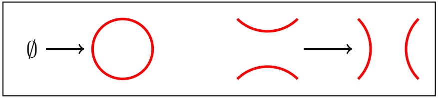
While the formula for functoriality in K-R's paper might look a little forbidding, we've actually already hit the heart of it.

The calculation for the unknot shows that the add or remove a cup functors are **biadjoint**.

one adjunction



other adjunction



Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$

Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$

Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$

Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$

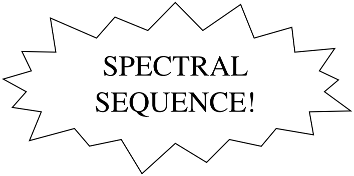
$$\begin{bmatrix} -x_1\otimes 1+1\otimes x_1 & x_2^N\otimes 1+\cdots+1\otimes x_2^N \\ x_2\otimes 1-1\otimes x_2 & -x_1^N\otimes 1-\cdots-1\otimes x_1^N \end{bmatrix}$$

$$\begin{array}{ccc} R\otimes_{R_i} R(-4) & \longrightarrow & R\otimes_{R_i} R(-2) \\ \oplus & & \oplus \\ R\otimes_{R_i} R & \longleftarrow & R\otimes_{R_i} R(-2) \end{array}$$

$$\begin{bmatrix} x_1\otimes 1-1\otimes x_1 & x_2\otimes 1-1\otimes x_2 \\ x_2^N\otimes 1+\cdots+1\otimes x_2^N & x_1^N\otimes 1+\cdots+1\otimes x_1^N \end{bmatrix}$$

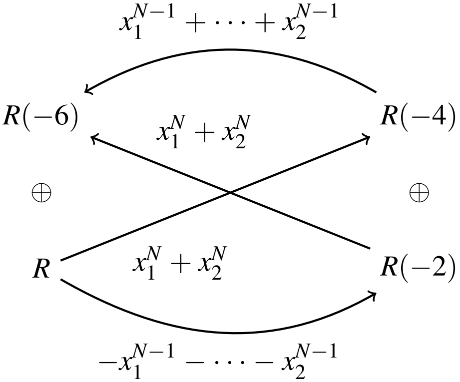
Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$



Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$



Examples

If we're going to do the computation for the Hopf link, we'll need to know the MF homology of $R \otimes_{R_1} R$ for $n = 2$

$$R(6)/(x_1^{N-1}+\cdots+x_2^{N-1}, x_1^N+x_2^N) \\ \cong H^*(\mathcal{F}\ell_{1,2}(\mathbb{C}^N))$$

The garden of good and evil

So what happened there? I used a spectral sequence which relates the MF homology to an iterated homology, where we use half the differentials (the "good" differentials) first, and the other half (the "evil" ones) on the result.

The garden of good and evil

So what happened there? I used a spectral sequence which relates the MF homology to an iterated homology, where we use half the differentials (the "good" differentials) first, and the other half (the "evil" ones) on the result.

Some noodling with homological algebra shows that

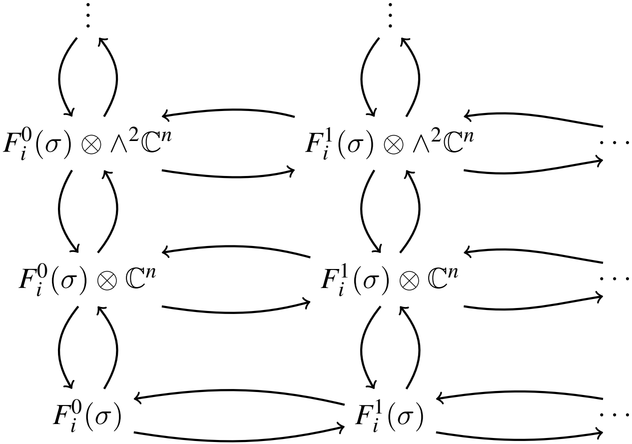
- If one replaces $F_i(\sigma)$ by a free resolution, one can make it into a MF with the right potential with any "evil" differentials we like, and this will not change either of these.
- We can then replace the matrix factorization deforming the resolution of the diagonal bimodule by the bimodule itself, and this will not change either of these.
- $F_i(\sigma)$ has a free resolution whose differentials are killed by $\otimes_{R\otimes R}R$.

Put these all together, and you get that the iterated homology is the same as the MF homology.

The garden of good and evil

$$
\begin{array}{c}
\vdots \\
\downarrow\uparrow \\
F_i(\sigma) \otimes \wedge^2 \mathbb{C}^n \\
\downarrow\uparrow \\
F_i(\sigma) \otimes \mathbb{C}^n \\
\downarrow\uparrow \\
F_i\sigma)
\end{array}
$$

The garden of good and evil



The garden of good and evil

$$HH_0(F_i(\sigma)) \longrightarrow HH_1(F_i(\sigma)) \longrightarrow \cdots$$

The garden of good and evil

Thus, the MF homology is really the homology of a strange differential on Hochschild homology!

$$HH_0(F_i(\sigma)) \longrightarrow HH_1(F_i(\sigma)) \longrightarrow \cdots$$

Exercise #1

Exercise #1

What is the MF homology of $R \otimes_{R_1} R \otimes_{R_2} R$ for arbitrary N and $n = 3$.

The Rasmussen spectral sequence

OK, now look at the whole complex for a knot.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & HH_{i+1}(F_j(\sigma)) & \longrightarrow & HH_{i+1}(F_{j+1}(\sigma)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & HH_{i}(F_j(\sigma)) & \longrightarrow & HH_{i}(F_{j+1}(\sigma)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

The Rasmussen spectral sequence

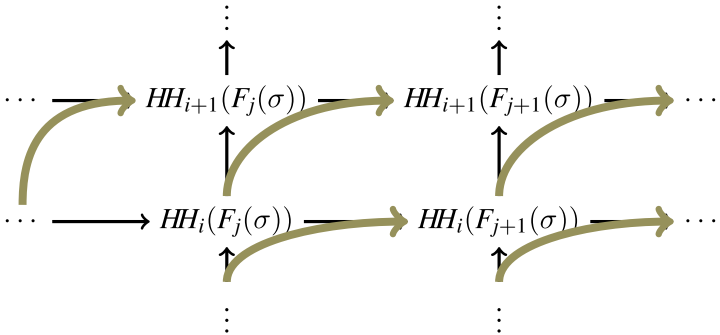
OK, now look at the whole complex for a knot.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & HH_{i+1}(F_j(\sigma)) & \longrightarrow & HH_{i+1}(F_{j+1}(\sigma)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & HH_i(F_j(\sigma)) & \longrightarrow & HH_i(F_{j+1}(\sigma)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

horizontal homology: HOMFLYPT

The Rasmussen spectral sequence

OK, now look at the whole complex for a knot.

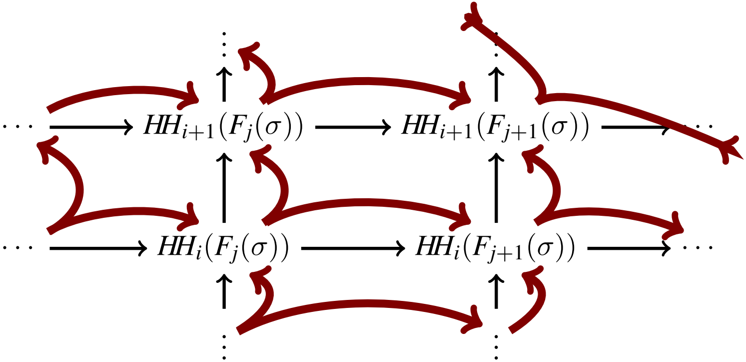


horizontal homology: HOMFLYPT

vertical then horizontal: K-R

The Rasmussen spectral sequence

OK, now look at the whole complex for a knot.



horizontal homology: HOMFLYPT

SS

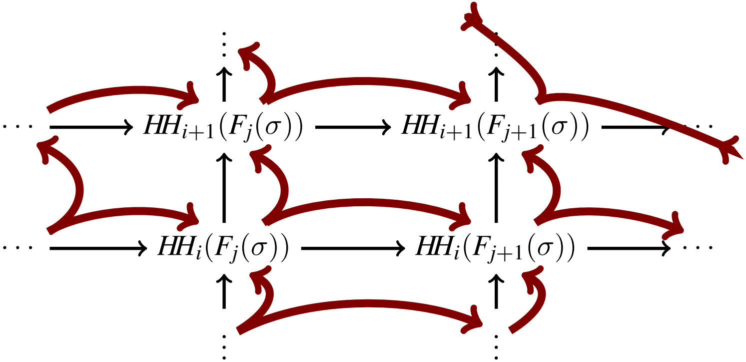
vertical then horizontal: K-R

SS

total homology: ?

The Rasmussen spectral sequence

OK, now look at the whole complex for a knot.



horizontal homology: HOMFLYPT

SS

vertical then horizontal: K-R

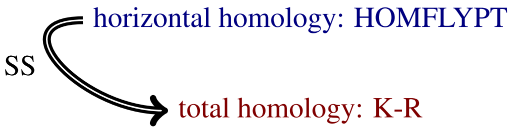
collapse!

total homology: K-R

The Rasmussen spectral sequence

OK, now look at the whole complex for a knot.

So we have a spectral sequence starting at HOMFLYPT, converging to K-R for each N.



Spectral sequences

Theorem (Rasmussen)

For all $N' \leq N$, there is a spectral sequence with E^2 page given byK-R homology for any N converging to K-R homology for N' (including the case of HOMFLY homology as $N = \infty$).

Spectral sequences

Theorem (Rasmussen)

For all $N' \leq N$, there is a spectral sequence with E^2 page given byK-R homology for any N converging to K-R homology for N' (including the case of HOMFLY homology as $N = \infty$).

Conjecture (Dunfield-Gukov-Rasmussen)

There is a spectral sequence with HOMFLYPT homology (or K-R homology for any $\mathfrak{gl}_N$) as E^2 page converging to knot Floer homology (which should be like K-R for $N = 0$)

At the moment, the only evidence I know of for this conjecture is computations by Rasmussen which show that in many cases such a spectral sequence can be constructed by hand. I don't think there is even a reasonable candidate for constructing it at the moment.

Examples

What about the Hopf link?

$$\begin{array}{c} R(-4) \\ \Big\downarrow \nu \\ R \otimes_{R_i} R(-2) \\ \Big\downarrow (x_1 - x_2) \otimes 1 + 1 \otimes (x_2 - x_1) \\ R \otimes_{R_i} R \end{array}$$

Examples

What about the Hopf link?

$$H^*(\mathbb{CP}^{N-1} \times \mathbb{CP}^{N-1})(-8)$$

$$\Big\downarrow \nu$$

$$H^*(\mathcal{F}\ell_{1,2}(\mathbb{C}^N))(-8)$$

$$H^*(\mathcal{F}\ell_{1,2}(\mathbb{C}^N))(-6)$$

Examples

What about the Hopf link?

$$H^*(\mathbb{CP}^{N-1})(-7-N) \oplus H^*(\mathbb{CP}^{N-1})(-6-N)$$

$$H^*(\mathbb{CP}^{N-1})(-8)$$

$$H^*(\mathcal{F}\ell_{1,2}(\mathbb{C}^N))(-6)$$

Examples

What about the Hopf link? What about the spectral sequence?

$$\begin{array}{c} \mathbb{C}[x_1](-2) \\ \downarrow \\ \mathbb{C}[x_1](-4) \oplus R \\ \downarrow \\ R(-2) \oplus R(-4) \\ \downarrow \\ R(-6) \end{array}$$

Examples

What about the Hopf link? What about the spectral sequence?

$$H^*(\mathbb{CP}^{N-1})(-7-N) \oplus H^*(\mathbb{CP}^{N-1})(-6-N)$$

$$H^*(\mathbb{CP}^{N-1})(-8)$$

$$H^*(\mathcal{F}\ell_{1,2}(\mathbb{C}^N))(-6)$$

Exercise #2

Exercise #2

What is the K-R homology of the trefoil? The figure 8 knot?