Diagrammatic categorification of quantum groups I

Aaron Lauda

Columbia University

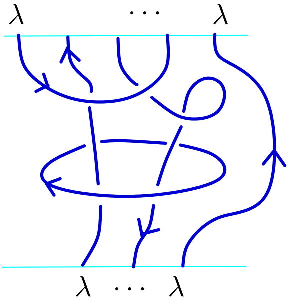
January 28th, 2010

Available at http://www.math.columbia.edu/∼lauda/talks/MSRI

Reshetikhin-Turaev invariant

For $\mathfrak{g}$ a simple Lie algebra the quantum deformation $U_q(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$ gives link/tangle invariants.

Colour the strands of a tangle by a representation V_λ of $U_q(\mathfrak{g})$



$V_\lambda^{\otimes n}$	$U_q(\mathfrak{g})$-module
$\uparrow \varphi(T)$	RT-invariant
$V_\lambda^{\otimes m}$	$U_q(\mathfrak{g})$-module

The invariant $\varphi(T)$ is a map of $U_q(\mathfrak{g})$-representations.

Example

- $\mathfrak{g} = \mathfrak{sl}_2$ — Jones polynomial, coloured Jones polynomial
- $\mathfrak{g} = \mathfrak{sl}_n$ — specializations of the HOMFLYPT polynomial

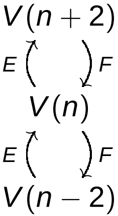
Quantum sl(2)

Definition

The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$ is the associative algebra (with unit) over $\mathbb{Q}(q)$ with generators E, F, K, K^{-1} and relations

- $KK^{-1} = 1 = K^{-1}K\,,$
- $KE = q^2 EK\,, \qquad KF = q^{-2}FK\,,$
- $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$

Consider a finite-dimensional representation V with a weight decomposition



$$V = \bigoplus_{n\in\mathbb{Z}} V(n)$$

$$Kv = q^n v\,, \qquad v \in V(n)$$

Why categorify quantum groups?

Conjectured applications to low-dimensional topology

- Representation theoretic explanation of Khovanov homology
- Categorification of the Reshetikhin-Turaev quantum knot invariants.
- Crane-Frenkel conjectured categorified quantum groups would give 4-dimensional TQFTs

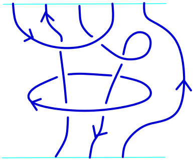
Categorified representation theory should provide new insights for ordinary representation theory.

- Geometric representation theory (algebraic/combinatorial analog of perverse sheaves)
- Positivity and integrality properties for quantum groups
- Representation theory of the symmetric group!

Quantum Group

representation
category

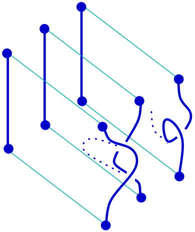
Braided monoidal
category with duals



Categorified
Quantum Group

representation
2-category

Braided monoidal
2-category with duals



Road map to categorification

There are several hints that suggest that quantum groups are really just shadows of a richer algebraic structure

- Lusztig's discovery of canonical basis for idempotent version $\dot{\mathbf{U}}$ of quantum groups.
- Geometric constructions of categorical quantum group actions
- The existence of semilinear forms $\langle , \rangle \colon \mathbf{U} \times \mathbf{U} \to \mathbb{Q}(q)$ with nice properties.

Beilinson, Lusztig, and MacPherson added orthogonal idempotents 1_n for projection onto $V(n)$ to produce $\dot{\mathbf{U}} := \dot{\mathbf{U}}_q(\mathfrak{sl}_2)$, a $\mathbb{Q}(q)$-algebra without unit

$$\begin{array}{ccc} \mathbf{U}_q(\mathfrak{sl}_2) & \longrightarrow & \dot{\mathbf{U}} \\ 1 & \longmapsto & \text{collection of orthogonal idempotents } 1_n \text{ for } n \in \mathbb{Z} \end{array}$$

$$K1_n = q^n 1_n \qquad \Rightarrow \text{no more } K$$

$$E1_n = 1_{n+2}E = 1_{n+2}E1_n \qquad\qquad F1_n = 1_{n-2}F = 1_{n-2}F1_n$$

$$EF1_n - FE1_n = [n]1_n \qquad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$\dot{\mathbf{U}}$ has a basis $\{E^aF^b1_n\}$ for $n \in \mathbb{Z}$, $a, b \geq 0$

Integral forms

The integral form $\dot{\mathbf{U}}_{\mathbb{Z}}$ of $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ is spanned by products of divided differences

$$E^{(a)}1_n := \frac{E^a}{[a]!}1_n \qquad F^{(b)}1_n := \frac{F^b}{[b]!}1_n$$

Crane and Frenkel conjectured (1994) that $\dot{\mathbf{U}}_{\mathbb{Z}}$ could be categorified using Lusztig's canonical basis

$$\begin{array}{ll} E^{(a)}F^{(b)}1_n, & n \leq b-a \\ F^{(b)}E^{(a)}1_n, & n \geq b-a \end{array}$$

Structure constants are in $\mathbb{N}[q,q^{-1}]$

$\dot{\mathbf{U}}_{\mathbb{Z}}$ is the Grothendieck ring of some higher structure $\dot{\mathcal{U}}$.

Why 2-categories?

$\dot{\mathbf{U}}$ is a nonunital algebra with a collection of mutually-orthogonal idempotents

(small) pre-additive categories $\longleftrightarrow$ **idempotented rings**

$\Rightarrow$ $\dot{\mathbf{U}}$ is a pre-additive category

- objects: $n \in \mathbb{Z}$
- morphisms $n \to m$: abelian group $1_m\dot{\mathbf{U}}1_n$
  - identities: 1_n
  - composition: $1_{m'}\dot{\mathbf{U}}1_m \otimes 1_{n'}\dot{\mathbf{U}}1_n \to \delta_{n',m}1_{m'}\dot{\mathbf{U}}1_n$ given by multiplication

$\Rightarrow$ the categorification $\dot{\mathcal{U}}$ of $\dot{\mathbf{U}}$ should be a 2-category

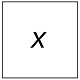
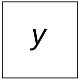
(small) pre-additive 2-categories $\longleftrightarrow$ **idempotented additive monoidal categories**

$\Downarrow$ Grothendieck category/ring

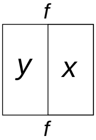
(small) pre-additive categories $\longleftrightarrow$ **idempotented rings**

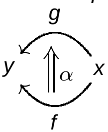
2-categories and string diagrams

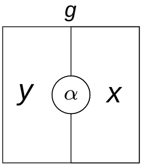
A 2-category is given by

- objects: represented by regions in the plane  or 

- morphisms: represented by lines separating regions of the plane

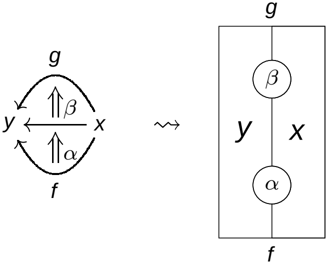
$y \xleftarrow{f} x \quad \rightsquigarrow$ 

- 2-morphisms: 

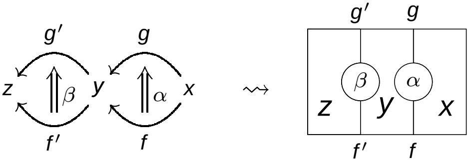
$\rightsquigarrow$ 

together with composition operations and identity 1 and 2-morphisms.

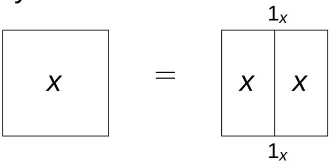
vertical composition

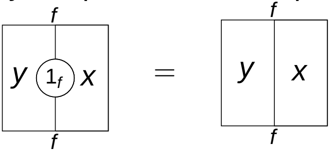


horizontal composition



By convention we do not draw identity morphisms or 2-morphisms:





Examples

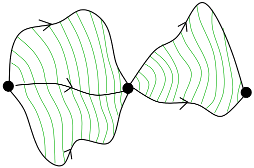
1. **Cat**:
   - objects: categories
   - morphisms: functors
   - 2-morphisms: natural transformations

1. **Bim**:
   - objects: commutative rings $R, S, T, \ldots$
   - morphisms: (S, R)-bimodules

     composition: $T \xleftarrow{{}_T N_S} S \xleftarrow{{}_S M_R} R \;:=\; T \xleftarrow{{}_T N_S \otimes_S {}_S M_R} R$
   - 2-morphisms: bimodule homomorphisms

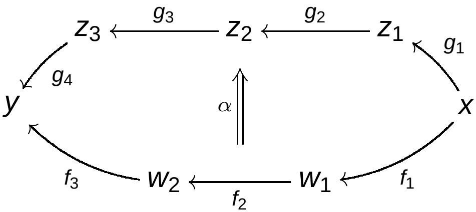
1. $\Pi(X)$:

   objects: points of a topological space X
   morphisms: paths in X
   2-morphisms: homotopies between paths

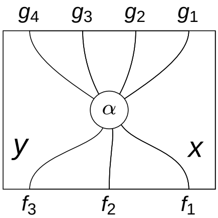


The last two examples are really *weak* 2-categories or bicategories.

If α is a 2-morphism



then α becomes the string diagram:

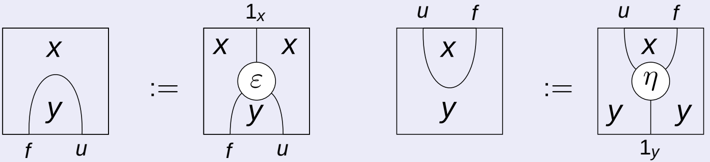


Now let's apply string diagrams to adjoint functors!

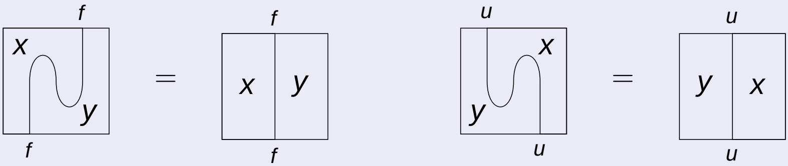
Definition

An adjunction in a 2-category consists of

- objects x and y
- morphisms $y \xleftarrow{f} x$ and $x \xleftarrow{u} y$
- 2-morphisms $1_x \Leftarrow f \circ u : \varepsilon$ and $u \circ f \Leftarrow 1_y : \eta$



such that the equalities



hold.

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_2) \quad \underset{\text{Grothendieck ring}}{\overset{\text{Categorification}}{\rightleftarrows}} \quad \dot{\mathcal{U}}_q(\mathfrak{sl}_2)$$

n weight	object n of $\mathcal{U}_q(\mathfrak{sl}_2)$
b basis element	1-morphism of $\mathcal{U}_q(\mathfrak{sl}_2)$
I.e., 1_n, $E1_n$, $F1_n$	$1_n, \quad \mathcal{E}1_n, \ \mathcal{F}1_n$
$q^a b$	$b\{a\} \ \Rightarrow$ (1-morphisms should be graded)
$x \cdot y = \sum_z m_{xy}^z z$	
m_{xy}^z structure constants in $\mathbb{N}[q, q^{-1}]$	$x \circ y = \bigoplus_z m_{xy}^z z$
	2-morphisms ???

Since the morphisms are graded, there is an internal 2-HOM given by taking homomorphisms of all degrees

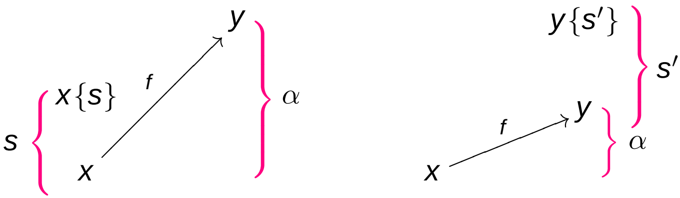
$$\mathrm{HOM}_{\dot{\mathcal{U}}}(\,,\,)\colon\ \mathrm{1morph}\left(\dot{\mathcal{U}}\right)\times\mathrm{1morph}\left(\dot{\mathcal{U}}\right)\longrightarrow \mathbf{GrVect}_k$$

Decategorification $\quad \rightsquigarrow K_0 \qquad \rightsquigarrow K_0 \qquad \rightsquigarrow \mathrm{gdim}$

$$\langle\,,\,\rangle\colon\ \dot{\mathbf{U}}\times\dot{\mathbf{U}}\longrightarrow\mathbb{Z}[q,q^{-1}]$$

Choice of 2-morphisms is controlled by such a form on $\dot{\mathbf{U}}$

This form will be semilinear (or sesquilinear) since $x \xrightarrow[\deg = \alpha]{f} y$



$$x\{s\} \longrightarrow y \qquad \deg = \alpha - s$$

$$x \longrightarrow y\{s'\} \qquad \deg = \alpha + s'$$

$$\langle q^s x, y \rangle = \mathrm{gdim}\left(\mathrm{HOM}(x\{s\}, y)\right) = q^{-s}\mathrm{gdim}\left(\mathrm{HOM}(x, y)\right) = q^{-s}\langle x, y \rangle$$

$$\langle x, q^{s'} y \rangle = \mathrm{gdim}\left(\mathrm{HOM}(x, y\{s'\})\right) = q^{s'}\mathrm{gdim}\left(\mathrm{HOM}(x, y)\right) = q^{s'}\langle x, y \rangle$$

From geometric considerations we expect that

$$\langle E^{(a)}1_n, E^{(a)}1_n\rangle = \mathrm{grdim} H^*(Gr(a,\infty)) = \prod_{j=1}^{a} \frac{1}{1-q^{2j}}$$

Example

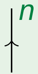
- $\langle E1_n, E1_n\rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$
- $\langle E^2 1_n, E^2 1_n\rangle = [2][2]\frac{1}{1-q^2}\frac{1}{1-q^4} = (1-q^{-2})(\frac{1}{1-q^2})$

Idea: use the semilinear form to guess generating 2-morphisms

Example

$$\mathrm{gdim}\left(\mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}1_n, \mathcal{E}1_n)\right) = \langle E1_n, E1_n \rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

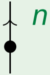
The identity 2-morphism $E1_n \Rightarrow E1_n$ must be degree zero

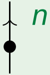
deg (

) $= 0$

contributes $q^0 = 1$ to graded dimension

The $q^2 \Rightarrow$ new 2-morphism $E1_n \Rightarrow E1_n$



with deg (

) $= 2$

contributes q^2 to graded dimension

Example (cont.)

Vertically composing the dot with itself we get

$$\deg\left(\overset{}{\underset{\alpha\bullet}{\uparrow}} \; n \right) := \deg\left(\left(n+2 \; \underset{\bullet}{\uparrow} \; n \right)^{\alpha} \right) = 2\alpha,$$

$$\mathrm{gdim}\left(\mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}1_n, \mathcal{E}1_n)\right)$$

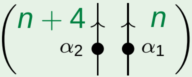
$$= \deg\left(\uparrow \; n \right) + \deg\left(\underset{\bullet}{\uparrow} \; n \right) + \deg\left(\underset{2\bullet}{\uparrow} \; n \right) + \cdots$$

$$= 1 + q^2 + q^4 + \cdots$$

$$= \frac{1}{1-q^2} = \langle E1_n, E1_n \rangle$$

Example ($\mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}^2 1_n, \mathcal{E}^2 1_n)$)

$$\sum_{0\leq\alpha_1, 0\leq\alpha_2} \deg \left(\text{diagram} \right) = (1+q^2+q^4+\dots)(1+q^2+\dots)$$

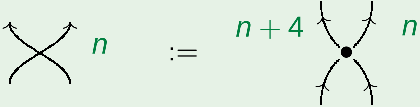


$$= \left(\frac{1}{1-q^2}\right)^2$$

The semilinear form gives $\mathrm{gdim}\, \dot{\mathcal{U}}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n)$

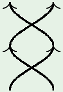
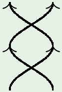
$$= \langle EE1_n, EE1_n \rangle = [2][2]\langle \mathcal{E}^{(2)}1_n, \mathcal{E}^{(2)}1_n \rangle = (1+q^{-2})\left(\frac{1}{1-q^2}\right)^2$$

$\Rightarrow$ an additional generating 2-morphism of degree -2:

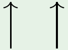
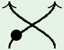
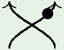


Example (cont.)

$$\operatorname{gdim} \dot{\mathcal{U}}(\mathcal{E}\mathcal{E}1_n, \mathcal{E}\mathcal{E}1_n) = (1-q^{-2})(1+q^2+q^4+\dots)(1+q^2+q^4+\dots)$$

deg ( n) $= -4$ $\Rightarrow$  n $= 0$

The space of degree zero 2-morphisms is 3-dimensional

$\Rightarrow$  n  n  n  n  n

are not linearly independent.

$\Rightarrow$ Add relations on these 2-morphisms

Considerations of adjoints in the geometric setting suggest that

$$\langle ux, y\rangle = \langle x, \tau(u)y\rangle \qquad \text{for } u \in \mathbf{U} \text{ and } x, y \in \dot{\mathbf{U}},$$

where $\tau \colon U \to U$ is such that

- $\tau(xy) = \tau(y)\tau(x)$ for all $x \in \dot{\mathbf{U}}$ (antihomomorphism)
- $\tau(1_n) = 1_n$
- $\tau(1_n E 1_{n-2}) = q^{1-n} 1_n F 1_{n+2}$
- $\tau(1_n F 1_{n+2}) = q^{1+n} 1_{n+2} E 1_n$

for all $n \in \mathbb{Z}$.

Example

$$\mathrm{gdim}\ \mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{F}\mathcal{E}1_n, 1_n) = \langle FE1_n, 1_n\rangle = \langle E1_n, \tau(F)1_n\rangle = \frac{q^{1+n}}{1-q^2}$$

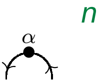
$$\mathrm{gdim}\ \mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}1_n, 1_n) = \langle EF1_n, 1_n\rangle = \langle F1_n, \tau(E)1_n\rangle = \frac{q^{1-n}}{1-q^2}$$

Add generators

generator		
degree	1+n	1-n

and dots so that

$$\mathrm{gdim}\ \mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{F}\mathcal{E}1_n, 1_n) = \mathrm{gdim} \sum_{\alpha=0}^{\infty} \Bigg($$



$$\Bigg)$$

$$= q^{1-n}(1 + q^2 + q^4 + \dots) = \frac{q^{1-n}}{1-q^2} = \langle FE1_n, 1_n \rangle$$

and

$$\mathrm{gdim}\ \mathrm{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}1_n, 1_n) = \mathrm{gdim} \sum_{\alpha=0}^{\infty} \Bigg($$

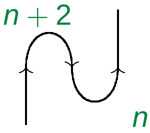
$$\Bigg)$$

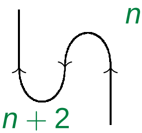
$$= q^{1-n}(1 + q^2 + q^4 + \dots) = \frac{q^{1-n}}{1-q^2} = \langle EF1_n, 1_n \rangle$$

Similar calculations for $\mathrm{HOM}_{\dot{\mathcal{U}}}(1_n, \mathcal{F}\mathcal{E}1_n)$ and $\mathrm{HOM}_{\dot{\mathcal{U}}}(1_n, \mathcal{E}\mathcal{F}1_n)$ suggest generators

generator		
degree	1+n	1-n

Using these 2-morphisms we can construct new 2-morphisms $\mathcal{E}1_n \Rightarrow \mathcal{E}1_n$ of the form

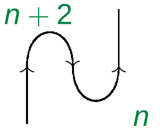


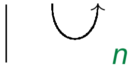


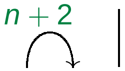
Recall that

$$\text{gdim}\left(\text{HOM}_{\dot{\mathcal{U}}}(\mathcal{E}1_n, \mathcal{E}1_n)\right) = \langle E1_n, E1_n \rangle = \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \dots$$

But

deg (

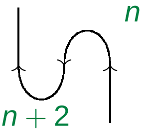
) = deg (

) + deg (

)

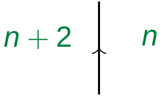
$$= 1 + n \quad + 1 - (n+2) \qquad = 0$$

similarly

deg (

) = 0

Hence these two 2-morphisms must be a multiple of the identity 2-morphism $\mathcal{E}1_n \Rightarrow \mathcal{E}1_n$



Definition

$\mathcal{U}$ is an additive $\Bbbk$-linear 2-category. The 2-category $\mathcal{U}$ consists of

- objects: n for $n \in \mathbb{Z}$.

The homs $\mathcal{U}(n, n')$ between two objects n, n' are additive $\Bbbk$-linear categories consisting of:

- objects of $\mathcal{U}(n, n')$: a 1-morphism in $\mathcal{U}$ from n to n' is a formal finite direct sum of 1-morphisms

  $$\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\} = 1_{n'}\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\}$$

  for any $s \in \mathbb{Z}$ and $n' = n + 2\sum \alpha_i - 2\sum \beta_i$.
- morphisms of $\mathcal{U}(n, n')$: the $\Bbbk$-vector space of 2-morphisms

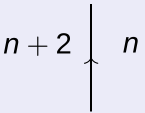
  $$\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\} \Longrightarrow \mathcal{E}^{\alpha'_1}\mathcal{F}^{\beta'_1}\cdots\mathcal{E}^{\alpha'_m}\mathcal{F}^{\beta'_m}1_n\{s'\}$$

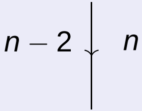
  given by linear combinations of degree $s - s'$ diagrams, modulo certain relations, built from composites of:

Generating 2-morphisms: identities

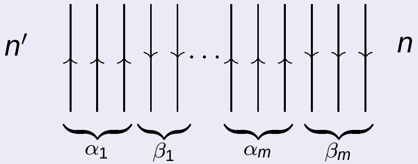
Degree zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}$; we write

$$1_{\mathcal{E}1_n\{s\}} \qquad\qquad 1_{\mathcal{F}\mathbf{1}_\lambda\{s\}}$$

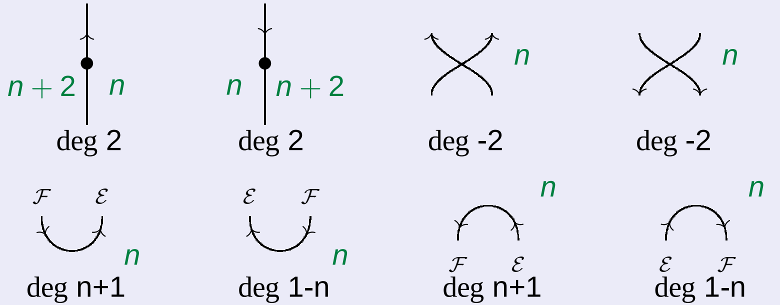




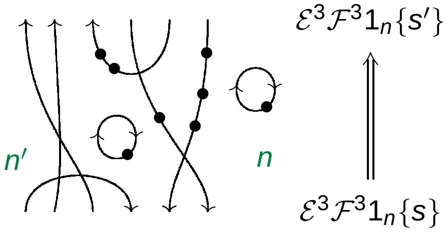
and more generally, the identity 2-morphism $1_{\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\}\mathbf{1}_\lambda\{s\}}$ is represented as



2-morphisms II



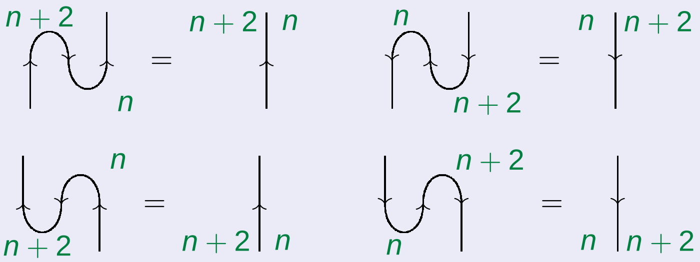
For example



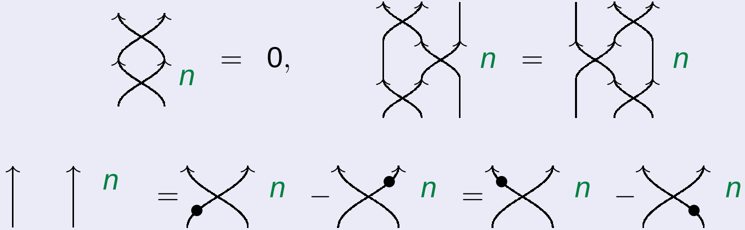
take degree $s - s'$ diagrams
this makes the total degree =0

Local relations

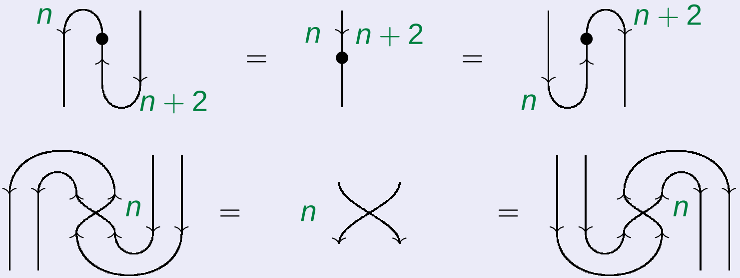
$\mathcal{E}1_n$ and $\mathcal{F}1_n$ are biadjoint up to grading shift



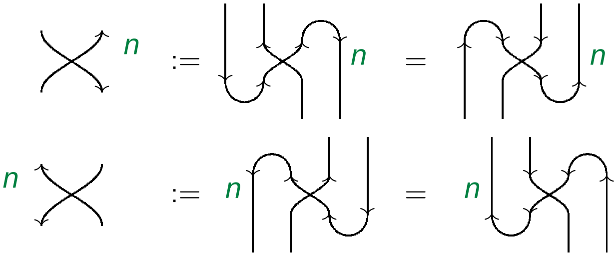
NilHecke relations



Topological invariance



We can define



Positivity of bubbles

All dotted bubbles of negative degree are zero. That is,

deg (

) $= 2(1-n) + 2\beta$

deg (

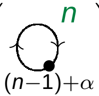
) $= 2(1+n) + 2\beta$

$\Rightarrow$ 

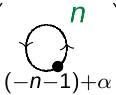
$= \quad 0 \quad$ if $\beta < n-1$

$= \quad 0 \quad$ if $\beta < -n-1$

It is convenient to emphasize the degree of a bubble on its label $\alpha \geq 0$

deg (

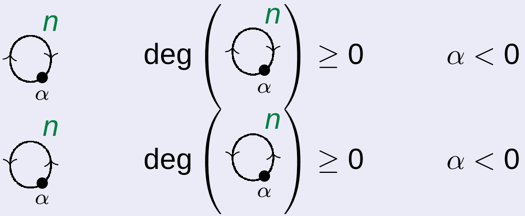
) $= 2\alpha$

deg (

) $= 2\alpha$

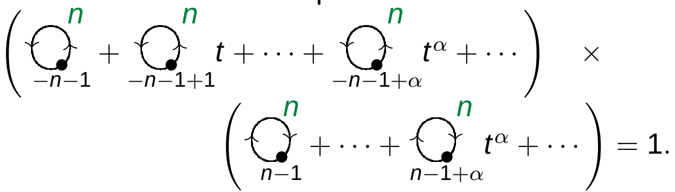
sometimes this is a negative number!

Fake bubbles



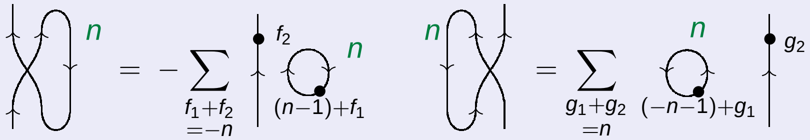
These formal symbols are inductively defined

Infinite Grassmannian equation:

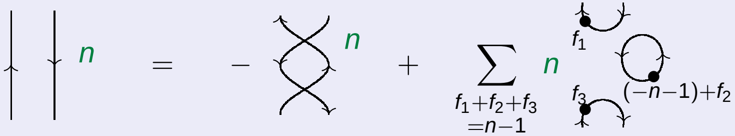


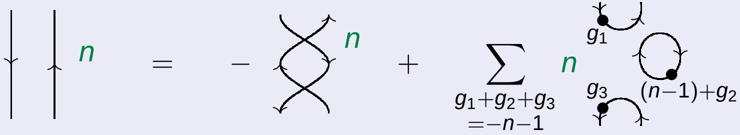
We set the degree zero bubbles equal to 1 for convenience.

Reduction to bubbles



EF decomposition



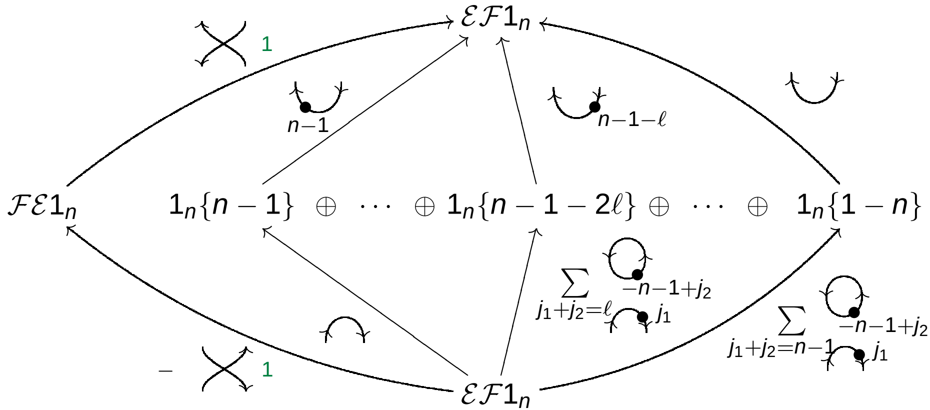


For $[n] = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$ write
$\oplus_{[n]} 1_n := 1_n\{n-1\} \oplus 1_n\{n-3\} \oplus \cdots \oplus 1_n\{1-n\}$

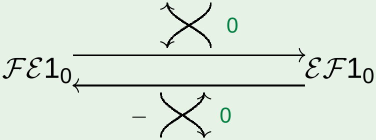
Lifting $\mathfrak{sl}_2$ relations

$\mathcal{E}\mathcal{F}1_n \cong \mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n$ for $n \geq 0$

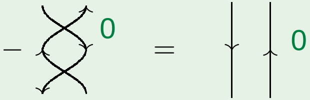
$\mathcal{F}\mathcal{E}1_n \cong \mathcal{E}\mathcal{F}1_n \oplus_{[-n]} 1_n$ for $n \leq 0$

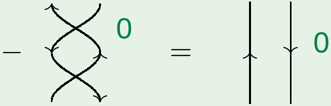


Example ($n = 0$, $\mathcal{E}\mathcal{F}1_0 \cong \mathcal{F}\mathcal{E}1_0$)



These maps are isomorphisms since for $n = 0$



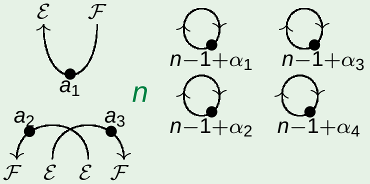


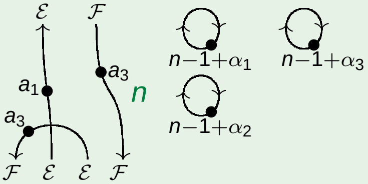
Spanning sets

Using relations one can find spanning sets for the space of 2-morphisms:

Example

$\mathcal{U}(\mathcal{F}\mathcal{E}^2\mathcal{F}1_n, \mathcal{E}\mathcal{F}1_n)$





- no strand intersects itself
- all closes diagrams are reduced to non-nested bubbles with the same orientation and are moved to the far right of each diagram
- all dots are confined to a small interval on each strand

Iterated flag varieties

$$\begin{array}{ccccc} & & Fl(k,k+1,N) & & \\ & \swarrow & \{0\subset\mathbb{C}^k\subset\mathbb{C}^{k+1}\subset\mathbb{C}^N\} & \searrow & \\ Gr(k,N) & & & & Gr(k+1,N) \\ \{0\subset\mathbb{C}^k\subset\mathbb{C}^N\} & & & & \{0\subset\mathbb{C}^{k+1}\subset\mathbb{C}^N\} \end{array}$$

$$\begin{array}{ccccc} & & H_{k,k+1} := H^*(Fl(k,k+1,N)) & & \\ & \nearrow & & \nwarrow & \\ H_k := H^*(Gr(k,N)) & & & & H_{k+1} := H^*(Gr(k+1,N)) \end{array}$$

$$H_N \cong \mathbb{C}$$

$$\vdots$$

$$H_{k+1}$$

$$H_{k+1,k} \quad \uparrow\!\!\downarrow \quad H_{k+1,k}$$

$$H_k$$

$$H_{k,k-1} \quad \uparrow\!\!\downarrow \quad H_{k,k-1}$$

$$H_{k-1}$$

$$\vdots$$

$$H_0 \cong \mathbb{C}$$

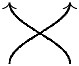
$$\Gamma_N : \mathcal{U} \longrightarrow \mathrm{Flag}_N$$

$$n \mapsto \begin{cases} H_k & n = 2k - N \\ \mathbf{0} & n \neq 2k - N \end{cases}$$

$$\mathcal{E}1_n \mapsto H_{k+1,k}$$

$$\mathcal{F}1_n \mapsto H_{k-1,k}$$

$$\mathcal{E}1_{n+2}\mathcal{E}1_n \mapsto H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k}$$

 $n \mapsto$

$$H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k}$$

$\Uparrow$ bimodule map

$$H_{k+2,k+1} \otimes_{H_{k+1}} H_{k+1,k}$$

Theorem

Γ_N is a 2-functor, all relations in $\mathcal{U}$ hold in the iterated flag category $\mathbf{Flag}_N$. Γ_N categorifies the irreducible $N+1$-dimensional representation.

Theorem (arXiv:0803.3652)

This graphical calculus is consistent and categorifies $\dot{\mathbf{U}}_{\mathbb{Z}}$

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category

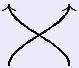
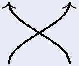
$$x \oplus y \in \dot{\mathcal{U}} \quad \rightsquigarrow \quad [x]+[y] \in K_0(\dot{\mathcal{U}}) \qquad x\{s\} \quad \rightsquigarrow \quad q^s[x] \in K_0(\dot{\mathcal{U}})$$

- Indecomposable 1-morphisms $\Leftrightarrow$ Lusztig canonical basis element
- Graded 2Hom $\mathrm{HOM}_{\dot{\mathcal{U}}}(x,y)$ categorifies the semilinear form $\langle x,y\rangle$
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N-dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$
- Various known (anti) linear (anti)automorphism of $\mathbf{U}_q(\mathfrak{sl}_2)$ have categorifications that are given on 2-morphisms by symmetries of the graphical calculus.

$$\begin{array}{rcl} \omega\colon \dot{\mathbf{U}} & \to & U \\ E1_n & \mapsto & F1_{-n} \\ F1_n & \mapsto & E1_{-n} \end{array} \quad \rightsquigarrow$$

Invert orientation

$n \mapsto -n$

 $n \mapsto -$  n

Joint with Mikhail Khovanov

- 2-category $\dot{\mathcal{U}}$ has an extension to a categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$.
- There is a categorification of $\dot{\mathbf{U}}^{+}(\mathfrak{g})$ for any Kac-Moody algebra $\mathfrak{g}$ using a similar diagrammatic calculus (arXiv:0803.4121, arXiv:0804.2080).
- Conjectural categorification of the integral form of $\dot{\mathbf{U}}(\mathfrak{g})$ for any Kac-Moody algebra (arXiv:0807.3250).

arxiv.0407205, arXiv:0812.5023

Closely related 2-categories were studied by Chuang and Rouquier.

arXiv:0902.1796

Geometric notions of categorical $\mathfrak{sl}_2$-actions and their connection with the 2-categories above have been studied by Cautis-Kamnitzer-Licata.