Knot homology for quantum invariants, through pictures

Ben Webster

MIT/Oregon

January 29, 2010

Quantum groups

Over the course of the 1980's various mathematicians (Drinfeld, Fadeev, Jimbo, Reshetikhin, etc.) came to the realization that universal enveloping algebras $U(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g}$ had interesting deformations as Hopf algebras.

While these have a lot of exciting properties and applications, they're still a bit hard to motivate when you first see them.

Quantum groups

Over the course of the 1980's various mathematicians (Drinfeld, Fadeev, Jimbo, Reshetikhin, etc.) came to the realization that universal enveloping algebras $U(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g}$ had interesting deformations as Hopf algebras.

While these have a lot of exciting properties and applications, they're still a bit hard to motivate when you first see them.

You can view this talk as my apologia for quantum groups based on one simple principle

"any interesting structure on a quantum group is probably the decategorification of something whose formula is easier to remember."

Decategorification?

For our purposes, decategorification means sending

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.
- an abelian category $\mathcal{C} \longrightarrow$ its Grothendieck group $K^0(\mathcal{C})$

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.
- an abelian category $\mathcal{C} \longrightarrow$ its Grothendieck group $K^0(\mathcal{C})$
- a graded abelian category $\mathcal{C} \longrightarrow$
  its $q-$Grothendieck group $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} K^0(\mathcal{C})$.

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.
- an abelian category $\mathcal{C} \longrightarrow$ its Grothendieck group $K^0(\mathcal{C})$
- a graded abelian category $\mathcal{C} \longrightarrow$
  its $q-$Grothendieck group $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} K^0(\mathcal{C})$.
- An exact functor $F : \mathcal{C} \to \mathcal{C}' \longrightarrow$
  the induced map $[F] : K^0(\mathcal{C}) \to K^0(\mathcal{C}')$.

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.
- an abelian category $\mathcal{C} \longrightarrow$ its Grothendieck group $K^0(\mathcal{C})$
- a graded abelian category $\mathcal{C} \longrightarrow$
  its $q-$Grothendieck group $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} K^0(\mathcal{C})$.
- An exact functor $F : \mathcal{C} \to \mathcal{C}' \longrightarrow$
  the induced map $[F] : K^0(\mathcal{C}) \to K^0(\mathcal{C}')$.

Decategorification?

For our purposes, decategorification means sending

- a vector space $V \longrightarrow$ its dimension $\dim V$.
- a graded vector space $V \longrightarrow$ its q-dimension $\dim_q V$.
- an abelian category $\mathcal{C} \longrightarrow$ its Grothendieck group $K^0(\mathcal{C})$
- a graded abelian category $\mathcal{C} \longrightarrow$
  its $q-$Grothendieck group $\mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} K^0(\mathcal{C})$.
- An exact functor $F : \mathcal{C} \to \mathcal{C}' \longrightarrow$
  the induced map $[F] : K^0(\mathcal{C}) \to K^0(\mathcal{C}')$.

So "$F : \mathcal{C} \to \mathcal{C}'$ categorifies $\phi : V \to V'$" means "there are isomorphisms $K^0(\mathcal{C}) \cong V$ and $K^0(\mathcal{C}') \cong V'$, such that the map induced by F is ϕ."

Pictorial categorification

One of the most successful programs of categorification has been the understanding of quantum groups, which goes back to Lusztig. Some remarkable progress on this story was made in the 80's and 90's. Using categorifications:

- Kazhdan and Lusztig defined a basis of the Hecke algebra.
- Lusztig defined the canonical basis of $U_q(\mathfrak{g})$ and its representations.

but this work at its heart was all geometric; there were a lot of perverse sheaves involved.

Pictorial categorification

One of the most successful programs of categorification has been the understanding of quantum groups, which goes back to Lusztig. Some remarkable progress on this story was made in the 80's and 90's. Using categorifications:

- Kazhdan and Lusztig defined a basis of the Hecke algebra.
- Lusztig defined the canonical basis of $U_q(\mathfrak{g})$ and its representations.

but this work at its heart was all geometric; there were a lot of perverse sheaves involved.

While geometry has a lot of power, it's also kinda hard. Luckily, in the past few years Rouquier and Khovanov-Lauda were able to redigest this whole story combinatorially, and so I can tell you an entirely pictorial story (though there's a still a little geometry tucked away in corners).

Tensor products

After what you've seen thus far, the logical next step is tensor products. Now, some of you might think: "wait, why can't you just take tensor product of the categories? That'll have the right Grothendieck group."

Tensor products

After what you've seen thus far, the logical next step is tensor products. Now, some of you might think: "wait, why can't you just take tensor product of the categories? That'll have the right Grothendieck group."

There are a couple of reasons why this is a bad idea. For one,

The canonical basis of a tensor product is not the tensor product of the canonical bases.

In fact, we'll see that there are objects categorifying the tensor product of the canonical bases, but they are **not** projective.

Tensor products

After what you've seen thus far, the logical next step is tensor products. Now, some of you might think: "wait, why can't you just take tensor product of the categories? That'll have the right Grothendieck group."

There are a couple of reasons why this is a bad idea. For one,

The canonical basis of a tensor product is not the tensor product of the canonical bases.

In fact, we'll see that there are objects categorifying the tensor product of the canonical bases, but they are **not** projective.

Secondly, the whole point of quantum groups is that they treat the two sides of the tensor product inequitably. We shouldn't expect a "democratic" construction, but one slanted toward one tensor factor or another.

Tensor products

Tensor products are worth doing correctly because of their importance in topology. I've been going on about the quantum group, but what I'm really after is its associated knot invariants, which arise from its structure as a ribbon category.

Tensor products

Tensor products are worth doing correctly because of their importance in topology. I've been going on about the quantum group, but what I'm really after is its associated knot invariants, which arise from its structure as a ribbon category.

Morally, these knot invariants arise from Chern-Simons theory. They "are" the expectation value of the trace on a chosen representation of the holonomy around the knot for a certain probability distribution on the space of $\mathfrak{g}$-connections on S^3.

This isn't actually a definition, unfortunately. (Unless you have some cool new ideas for making sense of path integrals.)

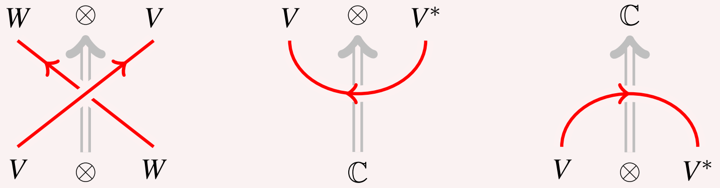
Reshetikhin-Turaev invariants

Reshetikhin and Turaev gave a mathematical construction of these invariants.

Reshetikhin-Turaev invariants

Reshetikhin and Turaev gave a mathematical construction of these invariants.

They label each component of the knot with a representation, and choose a projection of the knot. They then use the theory of quantum groups to attach maps to small diagrams like:

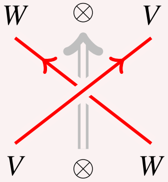


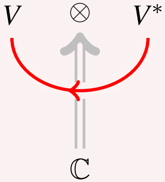
These are called the **braiding**, the **quantum trace** and the **coevaluation**.

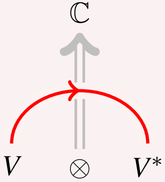
Reshetikhin-Turaev invariants

Reshetikhin and Turaev gave a mathematical construction of these invariants.

They label each component of the knot with a representation, and choose a projection of the knot. They then use the theory of quantum groups to attach maps to small diagrams like:







These are called the **braiding**, the **quantum trace** and the **coevaluation**.

Composing these together for a given link results in a scalar: the **Reshetikhin-Turaev invariant** for that labeling.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Cautis-Kamnitzer ('06): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Cautis-Kamnitzer ('06): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Khovanov-Rozansky('06): $\mathbb{C}^{2n}$ for $\mathfrak{so}_n$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Cautis-Kamnitzer ('06): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- Khovanov-Rozansky('06): $\mathbb{C}^{2n}$ for $\mathfrak{so}_n$.

A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while

- p Khovanov ('99): Jones polynomial ($\mathbb{C}^2$ for $\mathfrak{sl}_2$).
- ? Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $\mathfrak{gl}(1|1)$ invariant, and doesn't fit into our general picture).
- p Khovanov ('03): $\mathbb{C}^3$ for $\mathfrak{sl}_3$.
- c Khovanov-Rozansky ('04): $\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- p Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- c Cautis-Kamnitzer ('06): $\wedge^i\mathbb{C}^n$ for $\mathfrak{sl}_n$.
- c Khovanov-Rozansky('06): $\mathbb{C}^{2n}$ for $\mathfrak{so}_n$.

What I want to show is a unified, pictorial construction that should include all of these. p=proven, c=conjectured.

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

In particular, what we'd like to find is

- graded categories $\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}$ such that

$$K_q^0(\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

where V_{λ_i} is the representation of $U_q(\mathfrak{g})$ of highest weight λ.

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

In particular, what we'd like to find is

- graded categories $\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}$ such that

$$K_q^0(\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

  where V_{λ_i} is the representation of $U_q(\mathfrak{g})$ of highest weight λ.
- Graded functors categorifying:

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

In particular, what we'd like to find is

- graded categories $\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}$ such that

$$K_q^0(\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

  where V_{λ_i} is the representation of $U_q(\mathfrak{g})$ of highest weight λ.
- Graded functors categorifying:
  - the Chevalley generators E_i, F_i of $U_q(\mathfrak{g})$.

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

In particular, what we'd like to find is

- graded categories $\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}$ such that

$$K_q^0(\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

  where V_{λ_i} is the representation of $U_q(\mathfrak{g})$ of highest weight λ.
- Graded functors categorifying:
  - the Chevalley generators E_i, F_i of $U_q(\mathfrak{g})$.
  - the braiding maps relating different orderings of the weights of permutations.

Reshetikhin-Turaev invariants

The idea is to categorify everything in sight.

In particular, what we'd like to find is

- graded categories $\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}$ such that

$$K_q^0(\mathfrak{V}^{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

  where V_{λ_i} is the representation of $U_q(\mathfrak{g})$ of highest weight λ.
- Graded functors categorifying:
  - the Chevalley generators E_i, F_i of $U_q(\mathfrak{g})$.
  - the braiding maps relating different orderings of the weights of permutations.
  - the coevalution and quantum trace maps.

Tensor product algebras

Now we define an algebra generated by pictures consisting of

- red strands, colored with an $\mathfrak{g}$-rep
- black strands, colored with simple roots, each labeled with any number of dots

Tensor product algebras

Now we define an algebra generated by pictures consisting of

- red strands, colored with an $\mathfrak{g}$-rep
- black strands, colored with simple roots, each labeled with any number of dots

with the restrictions that

- strands must begin on $y = 0$, end on $y = 1$ and can never be horizontal
- red strands can never cross

Product is given by stacking (and is 0 if ends don't match).

Tensor product algebras

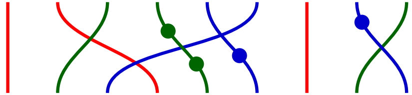
Now we define an algebra generated by pictures consisting of

- red strands, colored with an $\mathfrak{g}$-rep
- black strands, colored with simple roots, each labeled with any number of dots

with the restrictions that

- strands must begin on $y = 0$, end on $y = 1$ and can never be horizontal
- red strands can never cross

Product is given by stacking (and is 0 if ends don't match).

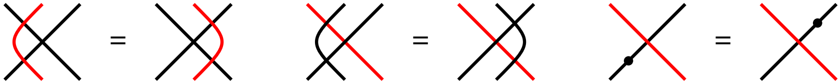


Tensor product algebras

First, we impose the Khovanov-Lauda relations from before. Also, we also need some relations involving red lines.

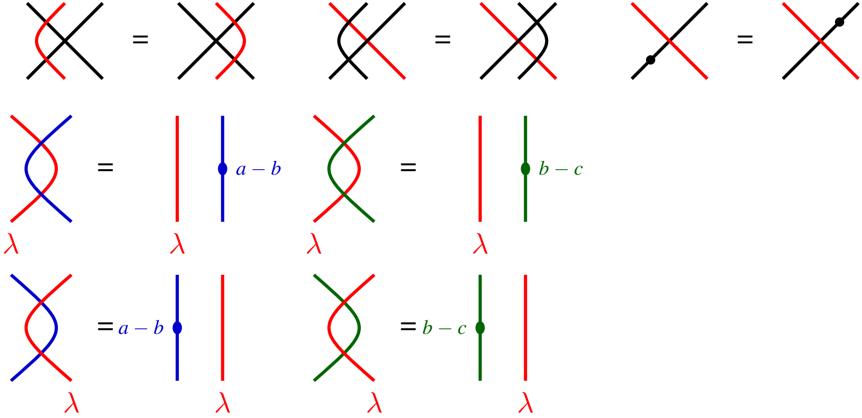
Tensor product algebras

First, we impose the Khovanov-Lauda relations from before. Also, we also need some relations involving red lines.



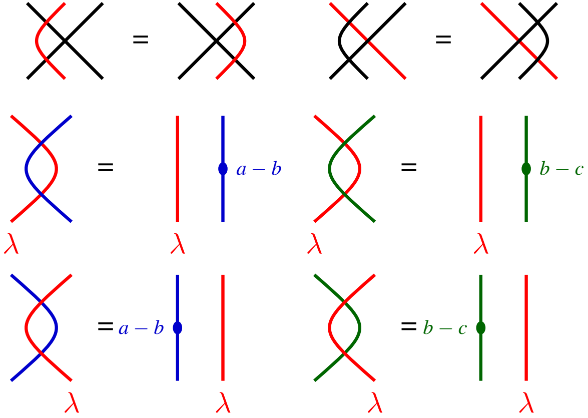
Tensor product algebras

First, we impose the Khovanov-Lauda relations from before. Also, we also need some relations involving red lines.



Tensor product algebras

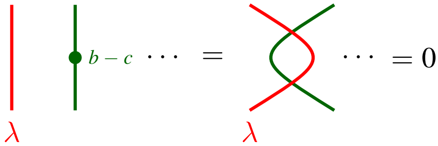
First, we impose the Khovanov-Lauda relations from before. Also, we also need some relations involving red lines.



Any diagram where a blue or green strand is to the left of all red strands is 0.

Comparison to Lauda-Vazirani

If there's only red line, then we only get one new interesting relation:



If there's only one red line labeled with λ, then we just get back E^{λ}, the cyclotomic quiver Hecke algebra.

Tensor product algebras

For a sequence of representations $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\ell)$, let $E^{\boldsymbol{\lambda}}$ be the subalgebra where the red lines are labeled with $\boldsymbol{\lambda}$ in order.

Theorem

$K(E^{\boldsymbol{\lambda}}\text{-mod}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} = V_{\boldsymbol{\lambda}}.$

Tensor product algebras

For a sequence of representations $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell)$, let $E^{\boldsymbol{\lambda}}$ be the subalgebra where the red lines are labeled with $\boldsymbol{\lambda}$ in order.

Theorem

$K(E^{\boldsymbol{\lambda}}\text{-mod}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} = V_{\boldsymbol{\lambda}}$.

There's a natural algebra map $I_1 : E^{\boldsymbol{\lambda}} \to E^{\boldsymbol{\lambda}}$ given by adding another blue strand, and similarly a map I_2 which adds a green strand.

Theorem

The action of E_1, E_2 is categorified by restriction of scalars by these maps, and F_1, F_2 by derived extension of scalars ($F \mapsto F \overset{L}{\otimes}_{E^{\boldsymbol{\lambda}}} E^{\boldsymbol{\lambda}}$) by these maps.

The derived category

So, I bet a lot of you are thinking “great, we’ll just look at the category of E^{λ}-modules.” Unfortunately, if you are, you’re wrong.

The functors we need that correspond to the braiding and trace of quantum group representations. There’s simply no exact functor which does that just on the level of the abelian category.

So we have to take some perspective which fixes the exactness, so something derived/dg/A_∞/stable $(\infty, 1)$ or something like that. For purposes of this talk, it will not really matter what you do.

The derived category

So, I bet a lot of you are thinking "great, we'll just look at the category of $E^{\boldsymbol{\lambda}}$-modules." Unfortunately, if you are, you're wrong.

The functors we need that correspond to the braiding and trace of quantum group representations. There's simply no exact functor which does that just on the level of the abelian category.

So we have to take some perspective which fixes the exactness, so something derived/dg/A_∞/stable $(\infty, 1)$ or something like that. For purposes of this talk, it will not really matter what you do.

I'll let $\mathcal{V}^{\boldsymbol{\lambda}} = D_{\mathsf{fd}}(E^{\boldsymbol{\lambda}}\text{-mod})$, the derived category where the sum of all cohomology objects is finite dimensional.

Note that $K(\mathcal{V}^{\boldsymbol{\lambda}}) \cong K(E^{\boldsymbol{\lambda}}\text{-mod})$ just by taking $[\mathbf{A}^\bullet] = \sum(-1)^i H^i(\mathbf{A}^\bullet)$.

Standard modules

The proof is by constructing a set of modules which categorify canonical bases. We call these **standard modules**. Consider the right ideal generated by all pictures where all red/black crossings are "negative."

negative crossing

positive crossing

Let $S^{\boldsymbol{\lambda}}$ denote the right $E^{\boldsymbol{\lambda}}$-module given by quotient by this ideal.

Standard modules

The proof is by constructing a set of modules which categorify canonical bases. We call these **standard modules**. Consider the right ideal generated by all pictures where all red/black crossings are "negative."


negative crossing


positive crossing

Let $S^{\boldsymbol{\lambda}}$ denote the right $E^{\boldsymbol{\lambda}}$-module given by quotient by this ideal.

Proposition

$\mathrm{End}(S^{\boldsymbol{\lambda}}) \cong E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell}$.

So, the failure of $S^{\boldsymbol{\lambda}}$ to be projective is exactly what encodes the difference between our tensor product category, and the naive tensor product.

Standard modules

We also obtain a functor from the naive product category to our category given by $- \overset{L}{\otimes}_{E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell}} S^{\boldsymbol{\lambda}} : \mathcal{V}^{\lambda_1;\ldots;\lambda_\ell} \to \mathcal{V}^{\boldsymbol{\lambda}}$.

Proposition

This functor induces $V_{\boldsymbol{\lambda}} \cong K(\mathcal{V}^{\lambda_1}) \otimes \cdots \otimes K(\mathcal{V}^{\lambda_\ell}) \cong K(\mathcal{V}^{\boldsymbol{\lambda}})$.

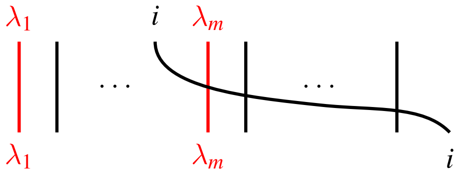
Standard modules

We also obtain a functor from the naive product category to our category given by $- \overset{L}{\otimes}_{E^{\lambda_1}\otimes\cdots\otimes E^{\lambda_\ell}} S^{\boldsymbol{\lambda}} : \mathcal{V}^{\lambda_1;\dots;\lambda_\ell} \to \mathcal{V}^{\boldsymbol{\lambda}}$.

Proposition

This functor induces $V_{\boldsymbol{\lambda}} \cong K(\mathcal{V}^{\lambda_1}) \otimes \cdots \otimes K(\mathcal{V}^{\lambda_\ell}) \cong K(\mathcal{V}^{\boldsymbol{\lambda}})$.

To see why this is so, consider $\mathfrak{F}_i(S^{\boldsymbol{\lambda}})$. This has a filtration given by elements



whose successive quotients match the terms of the coproduct

$$\Delta^{(\ell)}(F) = 1 \otimes \cdots \otimes F_i + \cdots + F_i \otimes K_i^{-1} \otimes \cdots \otimes K_i^{-1}.$$

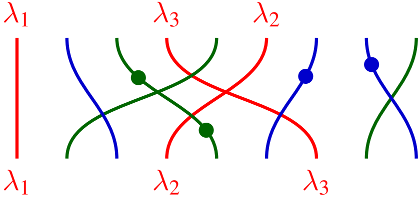
Braiding

So, now we need to look for braiding functors.

Braiding

So, now we need to look for braiding functors.

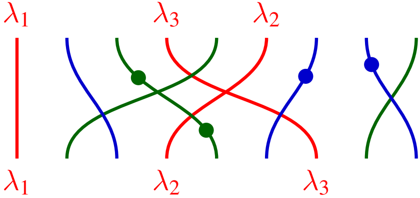
Consider the bimodule $\mathfrak{B}_i$ over $E^{\boldsymbol{\lambda}}$ and $E^{(i,i+1)\cdot\boldsymbol{\lambda}}$ given by exactly the same sort of diagrams, but with a single crossing inserted between the ith and $i+1$st crossings.



Braiding

So, now we need to look for braiding functors.

Consider the bimodule $\mathfrak{B}_i$ over $E^{\boldsymbol{\lambda}}$ and $E^{(i,i+1)\cdot\boldsymbol{\lambda}}$ given by exactly the same sort of diagrams, but with a single crossing inserted between the ith and $i+1$st crossings.



Theorem

The derived tensor product $-\overset{L}{\otimes}\mathfrak{B}_i : \mathcal{V}^{\boldsymbol{\lambda}} \to \mathcal{V}^{(i,i+1)\cdot\boldsymbol{\lambda}}$ *categorifies the braiding map* $R_i : V_{\boldsymbol{\lambda}} \to V_{(i,i+1)\cdot\boldsymbol{\lambda}}$. *The inverse functor is given by* $\mathrm{RHom}(\mathfrak{B}_i, -)$.

Coevalution and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights λ and $-w_0\lambda = \lambda^*$. In this case, pick a reduced expression

$$w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.$$

Coevalution and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights λ and $-w_0\lambda = \lambda^*$. In this case, pick a reduced expression

$$w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.$$

There's a unique simple module L_λ not killed by the idempotent for the sequence of roots $\alpha_1^{(\alpha_1^\vee(\lambda))}, \alpha_2^{(\alpha_2^\vee(s_1\lambda))}, \ldots, \alpha_n^{(\alpha_n^\vee(s_{n-1}\cdots s_1\lambda))}$.

Coevalution and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights λ and $-w_0\lambda = \lambda^*$. In this case, pick a reduced expression

$$w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.$$

There's a unique simple module L_λ not killed by the idempotent for the sequence of roots $\alpha_1^{(\alpha_1^\vee(\lambda))}, \alpha_2^{(\alpha_2^\vee(s_1\lambda))}, \dots, \alpha_n^{(\alpha_n^\vee(s_{n-1}\cdots s_1\lambda))}$.

- The coevaluation functor is categorified by the functor $\mathcal{V}^\emptyset \cong \mathsf{Vect} \to \mathcal{V}^{\lambda,\lambda^*}$ sending $\mathbb{C} \to L_\lambda$.
- The quantum trace functor is categorified by

$$\mathrm{RHom}(L_\lambda, -)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle) \colon \mathcal{V}^{\lambda,\lambda^*} \to \mathcal{V}^\emptyset \cong D_{\mathsf{fd}}(\mathsf{Vect}).$$

Coevalution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded dimension given by the quantum dimension of V_λ.

Coevalution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded dimension given by the quantum dimension of V_λ.

Has anyone seen this algebra before?

Coevalution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle)$$

has graded dimension given by the quantum dimension of V_λ.

Has anyone seen this algebra before?

One interesting candidate is the algebra structure that Feigin, Frenkel and Rybnikov put on V_λ using the "quantum shift of argument algebra" at a principle nilpotent.

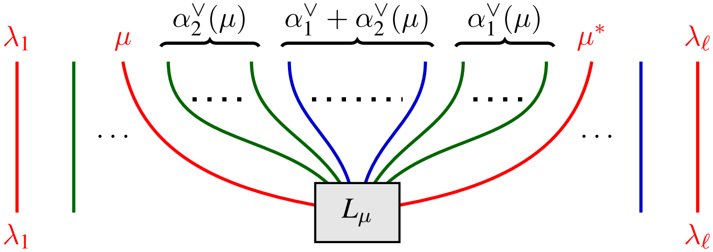
Another tantalizing possibility is that it is related to the geometry of $\overline{\mathrm{Gr}_\lambda}$. Perhaps a ring structure on intersection cohomology?

Coevalution and quantum trace

To do this in general, you can construct natural bimodules $\mathfrak{K}_\lambda$. Rather than give a definition, let me just draw the picture.

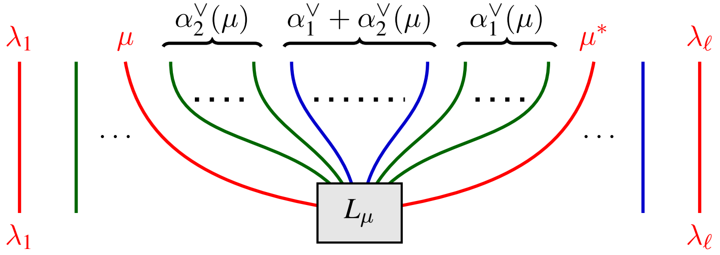
Coevalution and quantum trace

To do this in general, you can construct natural bimodules $\mathfrak{K}_\lambda$. Rather than give a definition, let me just draw the picture.



Coevalution and quantum trace

To do this in general, you can construct natural bimodules $\mathfrak{K}_\lambda$. Rather than give a definition, let me just draw the picture.

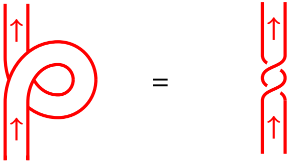


Theorem

Tensor product with this bimodule categorifies coevaluation, and Hom with it categorifies quantum trace.

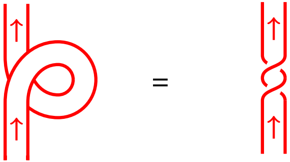
Ribbon structure

Now, I should have drawn all these pictures as ribbon knots, since framing matters in our picture. Moreover, I need to associate an actual functor to the ribbon twist.



Ribbon structure

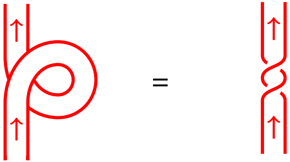
Now, I should have drawn all these pictures as ribbon knots, since framing matters in our picture. Moreover, I need to associate an actual functor to the ribbon twist.



The ribbon functor is just $M \mapsto M(\langle 2\lambda, \rho\rangle)[2\rho^\vee(\lambda)]$.

Ribbon structure

Now, I should have drawn all these pictures as ribbon knots, since framing matters in our picture. Moreover, I need to associate an actual functor to the ribbon twist.



The ribbon functor is just $M \mapsto M(\langle 2\lambda, \rho \rangle)[2\rho^\vee(\lambda)]$.

Note: this is a strange ribbon element! (It appeared in work of Snyder and Tingley on half-twist elements.) But that won't change things very much.

Knot invariants

Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.

Knot invariants

Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.

For a link L, we get a functor $F_L : \mathcal{V}^{\emptyset} \cong D_{\mathsf{fd}}(\mathsf{Vect}) \to \mathcal{V}^{\emptyset} \cong D_{\mathsf{fd}}(\mathsf{Vect})$. So $F_L(\mathbb{C})$ is a complex of vector spaces (actually graded vector spaces).

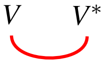
Theorem

The cohomology of $F_L(\mathbb{C})$ is a knot invariant. The graded Euler characteristic of this complex is $J_{V,L}(q)$.

Knot invariants

Start with $\mathbb{C}$.

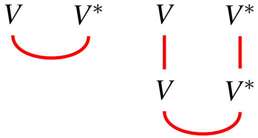
Knot invariants



$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ $\longrightarrow$ Replace with projective resolution B_1

$\longleftarrow$ Start with $\mathbb{C}$.

Knot invariants

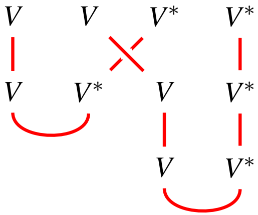


$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

Start with $\mathbb{C}$.

Knot invariants



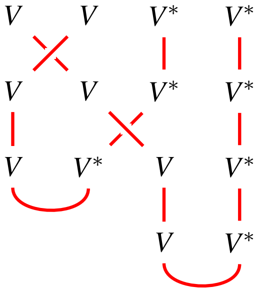
$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2)$ ⟶ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

Start with $\mathbb{C}$.

Knot invariants



$A_4 = B_3 \otimes \mathfrak{B}_1$ ⟶ Replace with projective resolution B_4

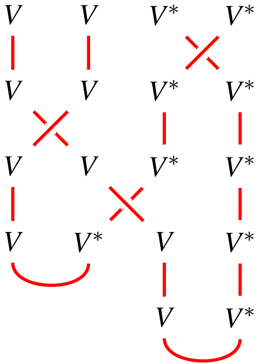
$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2)$ ⟶ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

Start with $\mathbb{C}$.

Knot invariants



$A_5 = B_4 \otimes \mathfrak{B}_3$ ⟶ Replace with injective resolution B_5

$A_4 = B_3 \otimes \mathfrak{B}_1$ ⟶ Replace with projective resolution B_4

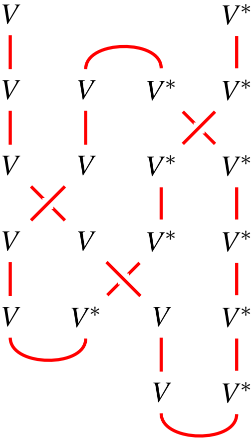
$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2)$ ⟶ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

Start with $\mathbb{C}$.

Knot invariants



$A_6 = \mathrm{RHom}(\mathfrak{K}_V^{2,3}, B_5) \longrightarrow$ Replace with injective resolution B_6

$A_5 = B_4 \otimes \mathfrak{B}_3 \longrightarrow$ Replace with injective resolution B_5

$A_4 = B_3 \otimes \mathfrak{B}_1 \longrightarrow$ Replace with projective resolution B_4

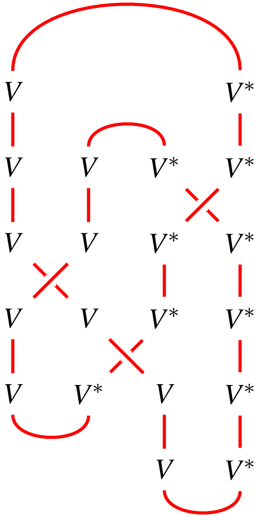
$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2) \longrightarrow$ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2} \longrightarrow$ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2} \longrightarrow$ Replace with projective resolution B_1

Start with $\mathbb{C}$.

Knot invariants



$A_7 = \mathrm{RHom}(\mathfrak{K}_V^{1,2}, B_6)$ ⟶ Knot homology!

$A_6 = \mathrm{RHom}(\mathfrak{K}_V^{2,3}, B_5)$ ⟶ Replace with injective resolution B_6

$A_5 = B_4 \otimes \mathfrak{B}_3$ ⟶ Replace with injective resolution B_5

$A_4 = B_3 \otimes \mathfrak{B}_1$ ⟶ Replace with projective resolution B_4

$A_3 = \mathrm{RHom}(\mathfrak{B}_i, B_2)$ ⟶ Replace with projective resolution B_3

$A_2 = B_1 \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with injective resolution B_2

$A_1 = \mathbb{C} \otimes \mathfrak{K}_V^{1,2}$ ⟶ Replace with projective resolution B_1

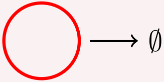
Start with $\mathbb{C}$.

Functoriality?

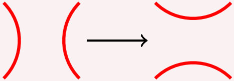
It's not known at the moment if this is functorial in cobordisms between knots. How would one construct a functoriality map?

Cobordisms of knots can be cut (using a Morse function) into the moves of

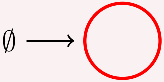
circle creation



saddle move



circle destruction

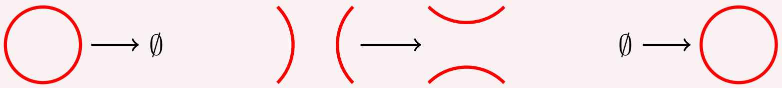


Functoriality?

It's not known at the moment if this is functorial in cobordisms between knots. How would one construct a functoriality map?

Cobordisms of knots can be cut (using a Morse function) into the moves of

circle creation — saddle move — circle destruction



Being able to define these maps requires that the cap and cup functors are *biadjoint* (they're clearly adjoint one way; the other one is quite non-trivial).

However, one has to prove that this map does not depend on the handle decomposition. Not easy!

4d TQFT

One of the inspirations for studying categorifications is the connections between higher categories and quantum field theory.

The quantum knot invariants arise from a 3-d TQFT: Chern-Simons theory. You can think of this as built up from attaching the category of $U_q(\mathfrak{g})$ representation to a circle and building the 2-and 3-dimensional layers from that.

4d TQFT

One of the inspirations for studying categorifications is the connections between higher categories and quantum field theory.

The quantum knot invariants arise from a 3-d TQFT: Chern-Simons theory. You can think of this as built up from attaching the category of $U_q(\mathfrak{g})$ representation to a circle and building the 2-and 3-dimensional layers from that.

Can one make a 4-dimensional TQFT of some kind out the category of 2-representations of this categorified quantum group?

Gukov and other physicists have done work on this, but as far as I know, nothing mathematically rigorous has appeared.

Comparison to classical representation theory

You might wonder if these are categories some of you have seen before? Well, probably some of you have:

Theorem

If $\mathfrak{g} = \mathfrak{sl}_n$ and $\boldsymbol{\lambda} = (\omega_1, \ldots, \omega_1)$, then $E^{\boldsymbol{\lambda}}$-mod is isomorphic is a direct sum of blocks of category $\mathcal{O}$ for $\mathfrak{sl}_k$ (which block one takes depends on n).

If $\boldsymbol{\lambda}$ is a sequence of fundamental weights then one must take parabolic category $\mathcal{O}$ for a parabolic whose block sizes are given by the indices of the fundamental weights.

Comparison to classical representation theory

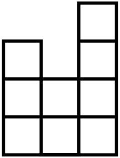
You might wonder if these are categories some of you have seen before? Well, probably some of you have:

Theorem

If $\mathfrak{g} = \mathfrak{sl}_n$ and $\boldsymbol{\lambda} = (\omega_1, \dots, \omega_1)$, then E^{λ}-mod is isomorphic is a direct sum of blocks of category $\mathcal{O}$ for $\mathfrak{sl}_k$ (which block one takes depends on n).

If $\boldsymbol{\lambda}$ is a sequence of fundamental weights then one must take parabolic category $\mathcal{O}$ for a parabolic whose block sizes are given by the indices of the fundamental weights.

For $\boldsymbol{\lambda} = (\omega_3, \omega_2, \omega_4)$, we take the category generated by the simples for column strict tableaux with entries in $[1, n]$ on the Young pyramid



Comparison to classical representation theory

You might wonder if these are categories some of you have seen before? Well, probably some of you have:

Theorem

If $\mathfrak{g} = \mathfrak{sl}_n$ and $\boldsymbol{\lambda} = (\omega_1, \dots, \omega_1)$, then E^{λ}-mod is isomorphic is a direct sum of blocks of category $\mathcal{O}$ for $\mathfrak{sl}_k$ (which block one takes depends on n).

If $\boldsymbol{\lambda}$ is a sequence of fundamental weights then one must take parabolic category $\mathcal{O}$ for a parabolic whose block sizes are given by the indices of the fundamental weights.

For $\boldsymbol{\lambda} = (\omega_3, \omega_2, \omega_4)$, we take the category generated by the simples for column strict tableaux with entries in $[1, n]$ on the Young pyramid

		9
5		8
3	7	4
2	1	1

Comparison to classical representation theory

If you want non-fundamental weights, you need to start taking subcategories generated by particular projective modules. Projectives correspond to column strict Young tableaux, so let me try to tell you which ones to take.

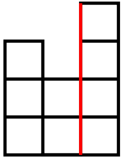
First write each weight $\lambda_i = a_{n-1}\omega_{n-1} + a_{n-2}\omega_{n-2} + \cdots$ and get a sequence of fundamental weights, with demarcated groups where the indices decrease and the groups sum to our desired weights.
The desired projectives are those that correspond to column strict tableaux on the Young pyramid which are semi-standard on each group.

Comparison to classical representation theory

If you want non-fundamental weights, you need to start taking subcategories generated by particular projective modules. Projectives correspond to column strict Young tableaux, so let me try to tell you which ones to take.

First write each weight $\lambda_i = a_{n-1}\omega_{n-1} + a_{n-2}\omega_{n-2} + \cdots$ and get a sequence of fundamental weights, with demarcated groups where the indices decrease and the groups sum to our desired weights.
The desired projectives are those that correspond to column strict tableaux on the Young pyramid which are semi-standard on each group.

For $\boldsymbol{\lambda} = (\omega_3 + \omega_2, \omega_4)$, we get the sequence $(\omega_3, \omega_2, \omega_4)$ and take the category generated by the projectives for column strict tableaux with entries in $[1, n]$ on the Young pyramid which are semi-standard on the first two rows:



Comparison to classical representation theory

If you want non-fundamental weights, you need to start taking subcategories generated by particular projective modules. Projectives correspond to column strict Young tableaux, so let me try to tell you which ones to take.

First write each weight $\lambda_i = a_{n-1}\omega_{n-1} + a_{n-2}\omega_{n-2} + \cdots$ and get a sequence of fundamental weights, with demarcated groups where the indices decrease and the groups sum to our desired weights.
The desired projectives are those that correspond to column strict tableaux on the Young pyramid which are semi-standard on each group.

For $\boldsymbol{\lambda} = (\omega_3 + \omega_2, \omega_4)$, we get the sequence $(\omega_3, \omega_2, \omega_4)$ and take the category generated by the projectives for column strict tableaux with entries in $[1, n]$ on the Young pyramid which are semi-standard on the first two rows:

		9
5		3
3	7	4
1	1	1

Open questions

- Can it be used practically to distinguish knots?

Open questions

- Can it be used practically to distinguish knots?
- Is there any way of doing computations efficiently?

Open questions

- Can it be used practically to distinguish knots?
- Is there any way of doing computations efficiently?
- What about Alexander polynomial? Could this prescription be modified to give Knot Floer homology?

Open questions

- Can it be used practically to distinguish knots?
- Is there any way of doing computations efficiently?
- What about Alexander polynomial? Could this prescription be modified to give Knot Floer homology?
- Clarify the geometric description.

Thanks, y’all.