Non-Locality in the Wavefunction of a Single Particle

If a wavefunction were a local disturbance, then detections at both places (or nowhere) could occur — would seem to imply superluminal communication.
Quantum-mechanical Spin

- for spin $\frac{1}{2}$
  - eg. electron, proton, neutron, quark, ...

Basis states

A: \[ \begin{align*}
A & : \quad \uparrow \\
B & : \quad \downarrow \\
wA + zB : & \quad \begin{cases}
\uparrow \downarrow & \text{for } w > z \\
\downarrow \uparrow & \text{for } w < z
\end{cases}
\end{align*} \]

Complex ratio: \[ \frac{z}{w} = u \]
The Celestial Sphere & the (past) light cone

**CONFORMAL property**

- Observer
- Light rays
- Stars

Differently moving observers perceive transformed celestial sphere but CIRCLES go to CIRCLES!
Twistor Theory

Minkowski Space-Time $M$

Twistor space
Quantum Geometry?

A commonly expressed view: Quantized metric $\rightarrow$ “fuzzy light cone”

Twistor view:

$\sim \sim \rightarrow$

“fuzzy point”
Twistor Theory

Minkowski Space-Time $M$:

$\begin{bmatrix} t, x, y, z \end{bmatrix}$

$c = 1$

Twistor space $Z^0 : Z^1 : Z^2 : Z^3$

**Incidence**

$\frac{i}{\sqrt{2}} \begin{bmatrix} 3 + t \ x + iy \\ x - it \ 3 - t \end{bmatrix} \begin{bmatrix} Z^2 \\ Z^3 \end{bmatrix}$

Eqn. of $PN$: $Z^0 \overline{Z^2} + Z^1 \overline{Z^3} + Z^2 \overline{Z^0} + Z^3 \overline{Z^1} = 0$
Twistor Theory

Minkowski Space-Time $\mathbb{M}$

Complex manifold

Rigidity of complex functions

Wavefunctions: Cohomology
Reality condition
\((r^0, r^1, r^2, r^3)\) real \(\Rightarrow\)
\[
\begin{pmatrix}
\overline{z^2} & \overline{z^3}
\end{pmatrix}
\begin{pmatrix}
z^0 \\
z^1
\end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix}
\overline{z^2} & \overline{z^3}
\end{pmatrix}
\begin{pmatrix}
r^0 + ir^3 & r^1 - ir^2 \\
r^1 + ir^2 & r^0 - ir^3
\end{pmatrix}
\begin{pmatrix}
z^2 \\
z^3
\end{pmatrix} = \text{pure imaginary}
\]

i.e.
\[z^0\overline{z^3} + z^1\overline{z^3} + z^2\overline{z^0} + z^3\overline{z^1} = 0\]

i.e.
\[\overline{Z_\alpha} Z_\alpha = 0\]

where
\[\overline{Z_\alpha} = (\overline{Z_0}, \overline{Z_1}, \overline{Z_2}, \overline{Z_3}) = (\overline{z^2}, \overline{z^3}, \overline{z^0}, \overline{z^1})\]

with \[Z_\alpha = (Z^0, Z^1, Z^2, Z^3)\]
Incidence: \( \omega^A = i r^{AA'} \pi_{A'} \)

Twistor: \( Z^\alpha = (\omega^A, \pi_{A'}) \)

Eqn. of PN: \( Z^\alpha \bar{Z}_{\alpha} = 0 \)

where \( \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}) \)

is a dual twistor

\[ \text{PT} = \mathbb{C}P^3 \]

Complexified space-time: \( \mathbb{C}M \)
allow \( r^{\alpha} \) to be complex.

Fix \( Z^\alpha \). Set of points of \( \mathbb{C}M \) incident with \( Z \) constitute an \( \alpha \)-plane. (For a dual twistor: \( \beta \)-plane)
\[ Z^2 = Z^3 = 0 \]
Conformally Compactified Minkowski Space $\mathbb{M}^#$

- $\mathcal{I}$
- $i^0$ → $i^+$
- Space
- $\mathfrak{p}^+$
- $\mathfrak{p}^-$

Minkowski space

- $w^2 + t^2 - x^2 - y^2 - z^2 - v^2 = 0$
- $ds^2 = dw^2 + dt^2 - dx^2 - dy^2 - dz^2 - dv^2$

Section by $w - v = 1$

$dt^2 - dx^2 - dy^2 - dz^2 = ds^2$

Minkowski space
Sophus Lie (1869)
Oriented spheres in $\mathbb{E}^3$

Complex

$\mathbb{C}P^3$

Minkowski 4-space

$\mathbb{E}^3$
### Twistor theory for different spacetime signatures

<table>
<thead>
<tr>
<th>Space-time signature</th>
<th>Twistor complex conjugation</th>
</tr>
</thead>
</table>
| ++ + + + + + (Atiyah, Hitchin, Singer) | $Z^\alpha \rightarrow \bar{Z}^\alpha$  
$W_\alpha \rightarrow \bar{W}_\alpha$  
no "real" twistors:  
$Z^\alpha = \bar{Z}^\alpha \Rightarrow Z^\alpha = 0$  
quaternionic case |
| ++ + + + + + (Duijanski, Mason, Witten) | $Z^\alpha \rightarrow \bar{Z}^\alpha$  
$W_\alpha \rightarrow \bar{W}_\alpha$  
real twistors $Z^\alpha = \bar{Z}^\alpha$  
give real vector 4-space  
$\mathbb{RP}^3 \subset \mathbb{CP}^3$  
real case |
| ++ + + + (or +++-) (Physical) | $Z^\alpha \rightarrow \bar{Z}^\alpha$  
$W_\alpha \rightarrow \bar{W}_\alpha$  
$Z^\alpha \bar{Z}_\alpha = 0$ gives light-ray space  
complex case |
Complex Minkowski Points

Imaginary part of complex position vector

Corresponds to forward tube of complex Minkowski space: past-timelike imaginary part

Positive-frequency fields: extend holomorphically to the forward tube composed of $e^{\frac{P_{\text{A}}}{2\sqrt{i}k}}$ with $P_{\text{A}}$: tails off in forward tube
\[ \frac{Z^a}{Z^b} = i \frac{r^0 + r^3}{r^1 - i r^2} \begin{pmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{pmatrix} \left( \frac{Z^2}{Z^3} \right) \]

\[ \omega^A = i r^{AA'} \pi_{A'} \]

\[ Z^\alpha = (\omega^A, \pi_{A'}) \]

**incidence:**  \[ \omega^A = i r^{AA'} \pi_{A'} \]

**shift of origin**

\[ \omega^A = \omega^A - i q^{AA'} \pi_{A'} \]

\[ \pi_{A'} = \pi_{A'} \]

**null twistor**

\[ Z^\alpha Z_\alpha = 0 \]

**equation of BPN**

**Momentum-angular mom. for massless ptcle.**

\[ P_a = 4\text{-momentum} \quad P_a P^a = 0 \quad (P_0 > 0) \]

\[ M^{ab} = 6\text{-angular momentum} \]

**Pauli-Lubanski spin vector**

\[ S_a = \frac{i}{2} \epsilon_{abcd} P^b M^{cd} = S P_a \]

**Note:** \((...)\) denotes symmetrization.
2-spinor Notation

Vector/tensor indices: $a, b, c, \ldots, a_0, \ldots, a_i, \ldots$
4-dimensional $0, 1, 2, 3$

$\triangleleft$ time

2-spinor indices:
$A, B, C, \ldots, A_0, \ldots, A_i, \ldots$
$A', B', C', \ldots, A'_0, \ldots, A'_i, \ldots$

Conjugate

Abstract indices:
$a = AA', \ b = BB', \ c = CC', \ldots, a_0 = A_0 A'_0, \ldots, a_i = A_i A'_i, \ldots$

Standard Coordinates:

\[
V^{AA'} : \begin{pmatrix} V^{00'} & V^{0i'} \\ V^{10'} & V^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}
\]

Raise/lower indices

$g_{ab}, g^{ab}$;\label{g}

Symmetric

$\varepsilon_{AB}, \varepsilon^{AB}, \varepsilon_{A'B'}, \varepsilon^{A'B'}$

Anti-symmetric

$g_{ab} = g^{AA' BB'} = \varepsilon_{AB} \varepsilon^{A'B'}$

Interpretation of $\eta^A$: \hspace{1cm} $\eta^A$
Quantum Twistor Theory

$Z^\alpha$ and $\bar{Z}_\alpha$ become non-commuting:

$$Z^\alpha Z^\beta - Z^\beta Z^\alpha = 0$$

$$\bar{Z}_\alpha \bar{Z}_\beta - \bar{Z}_\beta \bar{Z}_\alpha = 0$$

$$Z^\alpha \bar{Z}_\beta - \bar{Z}_\beta Z^\alpha = \hbar \delta^\alpha_\beta$$

So $Z^\alpha$ and $\bar{Z}_\alpha$ are canonical conjugate variables (as well as complex conjugate).

Choose $\hbar = 1$, for convenience. We find

$$P_a = \pi_A \bar{\pi}_{A'}$$

$$M^{ab} = i \omega^{(A'B')} \epsilon_{A'B'} - i \epsilon^{AB} \omega_{(A'B')}$$

undisturbed by factor ordering, but

$$S = \frac{1}{4} (Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha)$$

The standard commutators for $P_a$ and $M^{ab}$ follow

$[P_a, P_b] = 0$, $[P_a M^{bc}, M^{bd}] = i (g^{ac} p^e - g^{ae} p^b)$,

$[M^{ab}, M^{cd}] = i (g^{bc} M^{ad} - g^{bd} M^{ac} + g^{ad} M^{bc} - g^{ac} M^{bd})$.

Lie algebra generators for the Poincaré group.
Twistor Wavefunctions

Since $Z^\alpha$ and $(i)\bar{Z}_\alpha$ are conjugate variables, a twistor wavefunction $f$ ought to depend on either $Z^\alpha$ or $\bar{Z}_\alpha$, but not both. But what does it mean to say that $f(Z^\alpha)$ does not depend on $\bar{Z}_\alpha$? The condition is $\frac{\partial f}{\partial \bar{Z}_\alpha} = 0$, the Cauchy–Riemann equations asserting that $f$ is holomorphic in $Z^\alpha$.

Helicity eigenstates

These are eigenstates of the helicity operator $S = \frac{k}{2}(-2 - Z^\alpha \frac{\partial}{\partial Z^\alpha})$. But $Z^\alpha \frac{\partial}{\partial Z^\alpha}$ is the Euler homogeneity operator, whose eigenstates are homogeneous functions, with eigenvalue = degree of homogeneity.

Take the integer $n = 2S/k$ to represent the helicity; then $f$ is homogeneous (holomorphic) in $Z^\alpha$ of degree $-2 - n$. Alternatively use $f(W_\alpha)$, with $W_\alpha = Z_\alpha$. Then $f$ is hol. hom. deg. $-2 + n$. 
Free-field Maxwell equations

\[ F_{ab} = \psi_{ab}^\prime \varepsilon_{A'B'} + \varepsilon_{A'B'} \varphi_{A'B'} \]

implies

\[ F_{ab} = -F_{ba} \]

and

\[ \nabla^{A'A'} \psi_{AB} = 0, \quad \nabla^{A'A'} \varphi_{A'B'} = 0 \]

imply

\[ \nabla_{[a} F_{bc]} = 0, \quad \nabla^a F_{ab} = 0 \]

and conversely.

\[ dF = 0, \quad d^*F = 0 \]

Linearized source-free Einstein theory

\[ K_{abcd} = \psi_{ABCD}^\prime \varepsilon_{A'B'}^\prime \varepsilon_{c'd'} + \varepsilon_{AB} \varepsilon_{c'd'} \varphi_{A'B'C'D'} \]

implies

\[ K_{abcd} = -K_{bacd} = -K_{cdab} = K_{cdab} \]

and \[ K_{abcd} + K_{bcad} + K_{cabad} = 0, \ K_{abc} = 0; \]

moreover

\[ \nabla^{A'A'} \psi_{ABCD} = 0, \quad \nabla^{A'A'} \varphi_{A'B'C'D'} = 0 \]

imply (Bianchi)

\[ \nabla_a K_{bcde} + \nabla_b K_{cdae} + \nabla_c K_{dade} = 0 \]

and conversely.
### Massless field equations:

\[
\sum_{\frac{m}{n}} \phi_{AB\ldots L} = \phi_{(AB\ldots L)}, \quad \nabla^{A'A'} \phi_{AB\ldots L} = 0
\]

\[
\Box \phi = 0 \quad \text{helicity } -\frac{n}{2}
\]

\[
\phi_{A'B'\ldots L'} = \phi_{(A'B'\ldots L')}, \quad \nabla^{A'A'} \phi_{A'B'\ldots L'} = 0
\]

\[
\text{helicity } + \frac{n}{2}
\]

(assuming these are positive-frequency wave functions)

### Twistor function hom. deg.

<table>
<thead>
<tr>
<th>Helicity</th>
<th>Scalar wave</th>
<th>Dirac-Weyl</th>
<th>Maxwell photon</th>
<th>Linearized Einstein graviton</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Box equation (\Box \phi = 0)</td>
<td>Neutrino (\nabla^{A'A'} \gamma_A = 0)</td>
<td>Photon (\nabla^{A'A'} \phi_{A'B'} = 0)</td>
<td>Graviton (\nabla^{A'A'} \psi_{ABCD} = 0)</td>
</tr>
<tr>
<td>-2</td>
<td></td>
<td>Anti-neutrino (\nabla^{A'A'} \gamma_A = 0)</td>
<td>Left-handed (anti-self-dual) (\nabla^{A'A'} \phi_{A'B'} = 0)</td>
<td>Left-handed (anti-self-dual) (\nabla^{A'A'} \psi_{ABCD} = 0)</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td>Right-handed (self-dual) (\nabla^{A'A'} \phi_{A'B'} = 0, +\frac{1}{2})</td>
<td>Right-handed (self-dual) (\nabla^{A'A'} \psi_{ABCD} = 0, -\frac{1}{2})</td>
</tr>
<tr>
<td>-3</td>
<td></td>
<td></td>
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<tr>
<td>-4</td>
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<tr>
<td>-6</td>
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</tbody>
</table>
Contour Integral Expressions
(Whittaker, Bateman, RP, Hughston)

\[ \phi (x^a) = \text{const.} \oint f(\omega^A, \pi_A^a) \delta \pi \]
\[ \omega = i x \pi \]

Incidence

Homogeneous version (1-dim \( \phi \))
\[ \delta \pi = \pi_A^a d \pi_A^a \]

Inhomogeneous version (2-dim \( \phi \))
\[ \delta \pi = d \pi_A^a d \pi_B^a \]

Positive Helicity:
\[ \Phi_{A'B'...L'} (x^a) = \text{const.} \oint \pi_{A'} \pi_{B'} ... \pi_L f(\omega, \pi) \delta \pi \]
\[ \omega = i x \pi \]

Negative Helicity:
\[ \Psi_{AB...L} (x^a) = \text{const.} \oint \frac{\partial}{\partial \omega^A} \frac{\partial}{\partial \omega^B} ... \frac{\partial}{\partial \omega^L} f(\omega, \pi) \delta \pi \]
\[ \omega = i x \pi \]

Canonical case (elementary state):

\[ f = \frac{1}{(A \cdot Z^a)(B^a Z)^3} \]

General positive frequency:

\[ \mathbb{Z}^{\text{pos}} \]

PT

Riemann sphere

Lines in \( \mathbb{P}T^+ \) corr. pts. in the forward tube

P^T^+ 

Regions of Singularity

pts. with past-pointing imag. part

pts. with past-pointing imag. part
Massless Field Contour Integral

Hughston 1973, 1974

\[ S = 0 : \text{Wave eqn.} \quad \Box \phi = 0 \quad f(z) \text{ hom. deg.} = -2 \]

\[ \phi(x) = \oint f(z) \, dz \quad \omega = i \pi \]

Typical case:

\[ f(z) = \frac{1}{(A_{\alpha}z^\alpha)(B_{\beta}z^\beta)} \]

Riemann sphere \( x \)

 Generally:

\[ \nabla^{AA'} \phi_{A'B'C'...L'} = 0 \]

\[ \phi_{A'B'C'...L'}(x) = \oint \prod_{A'} \prod_{B'} \prod_{\pi_{L'}} f(z) \, dz \]

\[ \omega = i \pi \]

\[ \nabla^{AA'} \phi_{ABC...L} = 0 \]

\[ \phi_{AB...L}(x) = \oint \frac{\partial}{\partial \omega^A} \frac{\partial}{\partial \omega^B} ... \frac{\partial}{\partial \omega^L} f(z) \, dz \]

\[ S > 0 \]

\[ S < 0 \]
More generally, need

\[ f_{ij} = -f_{ji} \quad \text{on overlaps} \]

\[ f_{ij} + f_{jk} = f_{ik} \quad \text{on triple overlaps} \]

\[ f_{ij} \equiv f_{ij} + h_i - h_j \quad h_i \text{ def'd on } U_i \]

Branched contour $\mathcal{C}$

Non-linear (e.g. building a manifold)
Twistor (sheaf) cohomology

Cover (e.g.) $\mathbb{PT}^+$ with a (locally) finite number of open sets $\{U_i\}$

\begin{align*}
f_{ji} &= -f_{ij} \\
f_{ij} - f_{ik} + f_{jk} &= 0 \quad \text{on triple overlaps}
\end{align*}

\{f_{ij}\} is Čech cocycle

Branched contour integral evaluates \{f\} cocycle/coboundaries

\{f\} = \{g\} if \quad f_{ij} - g_{ij} = h_i - h_j \quad \text{on overlaps}

where $h_i$ defined (holomorphic) on $U_i$

$H'(\mathbb{PT}^+, \mathcal{O}(-25-2))$

massless tree fields

helicity $S$
Cohomology:
a precise non-local measure — here of the degree of IMPOSSIBILITY
Cohomology: a precise non-local measure — here of the degree of IMPOSSIBILITY
Positive/Negative Frequency Splitting

Riemann sphere $\mathbb{CP}^1$

Splitting of $H^0$ (ordinary functions) by holomorphic extension

$\text{real axis}$

Projective twistor space $\mathbb{PN} = \mathbb{CP}^3$

Splitting of $H^1$s, by holomorphic extension

$\mathbb{PN}$ (null twistors)

This realized (finally!) one of the very early motivations behind twistor theory.
Finite (Holomorphic) Deformations of Complex Manifolds — "Non-Linear 1st Sheaf Cohomology"

Cocycle vector field \( \{V_{ij}\} \) is infinitesimal form of finite deformation — produces genuinely distinct complex manifold, in many cases

Non-linear graviton Ward construction of ASD gauge fields
General Relativity

Numerous special applications (e.g. Woodhouse-Mason: stationary axi-symmetry)

As part of general programme: "non-linear graviton construction" [R.P. '76]

N.B. for flat space:

null separation  meeting lines

Deform:

R.P. 1976

null separation

General anti-self-dual complex "space-time"

General anti-self-dual Ricci-flat complex "space-time"
Googly (graviton)

A cricket ball bowled so as to look as though it spins left-handed whereas it actually spins right-handed.

Left-handed gravitons are described by anti-self-dual Weyl curvature, whereas right-handed gravitons are described by self-dual Weyl curvature.

Compare:
photons,
linear spin 2 massless quanta

Twistors, with usual conventions, had seemed to describe anti-self-dual interactions so far!
General relativity

Anti-self-dual curvature
(\textit{"non-linear graviton")}

$$F = \Gamma^\alpha_{\beta\gamma} \partial \phi_2 \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta}$$

\text{"infinity twistor"}

Gives general soln of asd. vacuum

Self-dual curvature—information coded (in strange way) by

$$G = f_\gamma Z^\alpha \frac{\partial}{\partial z^\alpha}$$

For asymptotically flat curved vacuum space-times, the full gravitational field is encoded geometrically (but strangely) in the structure of a deformed twistor space. Not yet clear how to retrieve the space-time from the twistor space.